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MATEMATIK ANALIZDAN QISQA KURS

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SO'Z BOSHI

Matematik analiz fanidan juda ko'p darsliklar mavjud bo'lib, ular bir necha jilddan (qismdan) iborat va ularning ko'pchiligi krill alifbosida yoki rus tilida yozilgan. Bu kamchiliklarni to'ldirishda T.Azlarov va H.Mansurov tomonidan yozilgan ikki jildlik kitob, so'ng bu kitobni bakalavr uchun yozilgan variantlari muhim o'rin tutadi. Shu bilan birga bu kitoblarga amaliy mashg'ulotlarga bo'lgan ehtiyoj mualliflar A.Sadullayev, H.Mansurov, G.Xudoyberganov, A.Vorisov, R.G'ulomov tomonidan tayyorlangan "Matematik analiz kursidan misol va masalalar to'plami" nomli ikki jildli qo'llanma yordamida to'ldirildi. Bular matematika ixtisosligi bo'yicha mutaxassislar tayyorlash dasturi asosida yozilgan. Matematik analiz fani soatlar soni qisqartirilgan holda boshqa mutaxassisliklarda ham o'qitiladi. Oliy matematika, hisob fanlarida matematik analiz mavzulari qisqa holda o'qitiladi. Shu bilan birga oliy o'quv yurtlarining sirtqi bo'limining ko'pgina mutaxassisliklarida matematik analiz qisqartirilgan soatlarda o'qitiladi.

Mutaxassislik olayotga talabalar uchun matematik analizning asosiy mavzularini o'zlashtirish va matematikaning (boshqa fanlarning) boshqa bo'limlarini uzluksiz o'zlashtirish uchun doimo qo'llaniladigan mavzularni o'z ichiga olgan va qisqa yozilgan kitob bo'lgani maqsadga muvofiq deb o'ylaymiz.

Ushbu darslikda analiz fanini o'rganish uchun kerak bo'ladigan barcha boshlang'ich tushunchalar, belgilarning kiritilish tarixi, aosiy ta'rif va teoremalar, mavzularga doir misollar izohlar bilan yechib ko'rsatilgan. Har bir mavzu so'ngida testlar va mustaqil bajarish uchun misollar keltirilgan. Ulardan nazoratlar va sirtqi bo'lim talabalari uchun yozma topshiriqlar tayyorlashda foydalanish mumkin.

Darslik didaktik asosda yozilgan va kerakli "mavzu" (tushuncha) mundarija orqali oson topiladi.

Kitob oxirida asosiy funksiyalar grafiklari va elementar matematikaga doir barcha asosiy formulalar keltirilgan.

Ushbu kitob matematika, oliy matematika va matematik analiz fanlari o'qitiladigan oliy o'quv yurtlari talabalari uchun mo'ljallangan. Qo'lyozmani sinchiklab o'qib, uni metodik jihatdan yaxshilanishiga hissalarini qoshganlilari uchun O'zbekiston milliy universiteti

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Agar siz o‘quvchilar ham qo‘llanmaning kamchilik va xatoliklari haqida o‘z fikrlaringizni bildirsangiz biz o‘z minnatdorchiligimizni bildirgan bo‘lar edik.

Mualliflar

1§. AYRIM ASOSIY MATEMATIK TUSHUNCHALAR

Biror tasdiq hamda “rost” yoki “yolg‘on” mulohaza yuritish mumkin bo‘lsa, u “***fikr***” (*gap, ma’noli so‘z, bayon*) deyiladi. Fikrlar A, B, C va hokazo lotin alifbosining bosh harflari bilan belgilanadi. Masalan, $A \equiv \{5 < 7\}$, $B \equiv \{Toshkent \quad O'zbekiston\ poytaxti\}$, $C \equiv \{9, \quad tub\ son\}$. Bu misolda A, B ***fikrlar*** “rost”, C-esa “yolg‘on” fikrlar.

Sodda “fikrlardan” murakkab fikrlar “va”, “yoki”, “faqat va faqat”, “agar…, u holda …” so‘zlari yordamida tashkil qilinadi. Fikr bir yoki bir necha parametrga bog‘liq bo‘lishi mumkin va ular $A(n)$, $B(n, m)$, $C(x, y, z)$,… kabi belgilanadi. Fikr faqat biror sohada ma’noga ega bo‘lishi mumkin. Shu sababli u qaysi sohada berilgani ko‘rsatilishi kerak. Masalan, $A(x) = \{x^2 + 1 > 0\}$, $x \in R$ da “to‘g‘ri”.

Har bir fan boshlang‘ich tushuncha, tasdiq va aksiomalar asosida quriladi.

Aksioma (***grekcha so‘z bo‘lib-qabul qilingan holat, talab qilaman*** degan ma’noni bildiradi)-to‘g‘riligi o‘z-o‘zidan ko‘rinib turgan va isbotsiz qabul qilingan “fikrlar”ga aytiladi.

Lemma (***grekcha so‘z bo‘lib, faraz, taxmin***)-boshqa tasdiqlarni (teoremalarni) isbot qilishda ishlatiladigan yordamchi “tasdiq”.

Teorema (***grekcha so‘z bo‘lib, qarayman, o‘****rganaman degan ma’noni bildiradi*)-matematik tasdiq bo‘lib, to‘g‘riligi isbot qilish bilan ko‘rsatiladi.

Teorema asosan ikki qismdan iborat bo‘ladi: “shart” va “xulosa”dan iborat. Teoremaning “shart” qismi, “xulosa” qismidan “u holda” (“unda”) so‘zi bilan ajraladi.

Misol. 1-**teorema.** Agar uchburchak burchaklaridan bittasi o‘tmas bo‘lsa, u holda qolgan ikkitasi o‘tkir bo‘ladi.

Bu teoremada “Agar uchburchak burchaklaridan bittasi o‘tmas bo‘lsa” so‘zlari teoremaning “shart” qismi, qolgan qismi “xulosa” bo‘ladi. Qisqacha teorema tuzilishini “agar …, u holda …” ko‘rinishida tasvirlash mumkin.

Teoremani quyidagi “fikr” (bayon) ko‘rinishida yozish mumkin:

$$(\forall x)\ A(x) \Rightarrow B(x),\ x \in M\ .$$

Har qanday “agar …, u holda …” ko‘rinishida ifodalangan teoremani unga ***teskari teorema*** holda ifodalash mumkin. Berilgan teoremaning "sharti" uning "xulosasi" (natijasi) bilan, "xulosasi" esa "sharti" bilan almashtirilsa, unga ***teskari teorema*** hosil bo'ladi. Bu holda berilgan teorema to'g'ri, ikkinchisi esa teskari teorema bo'ladi.

To'g'ri va unga teskari teoremalar ***o'zaro teskari teoremalar*** deyiladi. Masalan, quyidagi 2- teoremaga qaraylik. Agar berilgan natural sonning raqamlar yig'indisi 9 ga bo'linsa, u holda u son 9 ga bo'linadi degan teoremani "to'g'ri" teorema desak, uni teskarisi "Agar berilgan son 9 ga bo'linsa, u holda sonning raqamlar yig'indisi 9 ga bo'linadi" kabi ifodalanadi.

Agar $A(n) = \{n\ sonining\ raqamlar\ yig'indisi\ 9\ ga\ bo'linadi\}$, $B(n) = \{n\ soni\ 9\ ga\ bo'linadi\}$ va bularda $n \in N$ "fikr"lardan iborat bo'lsin.

Agar teorema

$$(\forall n) \quad A(n) \Rightarrow B(n), n \in N \text{ va } (\forall n) \quad B(n) \Rightarrow A(n), n \in N$$

ko'rinishida bo'lsa, u ***o'zaro teskari*** bo'ladi. Teskari teorema har doim o'rinli bo'lmasligi mumkin.

1-teoremaning teskarisi o'rinli emas, ya'ni "Agar biror uchburchakning ikki burchagi o'tkir bo'lsa, u holda u to'g'ri burchakli bo'ladi" degan teorema o'rinli emas.

To'g'ri va teskari teoremalar o'zaro teskari bo'lishi uchun, ulardagi shartlar na faqat ***yetarli*** bo'lishi, shu bilan xulosa o'rinli bo'lishi uchun ***zarur*** ham bo'lishi kerak.

"To'g'ri" va "teskari" teoremalar bilan "zarur" va "yetarli" hamda "zaruriy va yetarli" so'zlar chambarchas bog'liq.

Agar $(\forall x)$ $A(x) \Rightarrow B(x)$ teorema o'rinli bo'lsa, $A(x)$ "fikr" $B(x)$ uchun ***yetarli shart***, $B(x)$ "fikr" esa $A(x)$ uchun ***zaruriy shart*** deyiladi.

Bularni farqlash uchun quyidagi teoremani qaraylik.

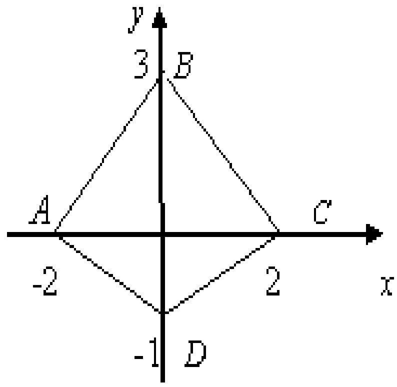
3-teorema. Agar to'rtburchak to'g'ri burchakli bo'lsa, u holda uning diagonallari o'zaro teng bo'ladi.

Q to'plam x-to'rtburchaklardan iborat bo'lsin, u holda $A(x) \equiv \{x\ to'rtburchaklar\ to'g'ri\ to'rburchak\}$, $B(x) \equiv \{x\ to'rtburchak\ diagonallari\ o'zaro\ teng\}$ "fikrlar" bo'lsin.

3-teorema to'g'ri, shu sababli $A(x)$ shart $B(x)$ uchun ***yetarli***, ya'ni to'rtburchakning diagonallari teng bo'lishi uchun uning to'g'ri burchakli bo'lishi ***yetarli***. $B(x)$ "fikr" ("natija", "xulosa") $A(x)$ uchun ***zaruriy shart*** bo'ladi, chunki to'rtburchak to'g'ri burchakli bo'lishi uchun uning diagonallari teng bo'lishi ***zaruriy shart*** (yetarli emas).

Diagonallar tengligidan to‘rtburchakning doimo to‘g‘ri to‘rtburchak bo‘lishi kelib chiqavermaydi (1-rasmga qarang).

Agar $(\forall x)\ A(x)\Rightarrow B(x),\, x\in M$ teorema o‘rinli bo‘lishi bilan unga teskari teorema $(\forall x)\ B(x)\Rightarrow A(x),\, x\in M$ teskari teorema ham o‘rinli bo‘lsa, $A(x)$ “shart” $B(x)$ uchun ***“zarur” va “yetarli”*** shart, $B(x)$ esa $A(x)$ uchun zaruriy va yetrli shart bo‘ladi (deyiladi). Bunga misol qilib 2-teoremani qarash mumkin, ya’ni “sonning 9 ga bo‘linishi uchun, uning raqamlar yig‘indisi 9 ga bo‘linishi zarur va yetarli”.



1- rasm

Eslatma. Agar teoremada “zaruriy va yetarli” shartlar keltirilgan bo‘lsa, uning isboti zaruriyligi hamda yetarli isbotlab ko‘rsatilishi shart.

Ayrim hollarda “zaruriy va yetarli” so‘zlar o‘rnida “faqat va faqat shunda” kabi so‘zlar ishlatilishi mumkin.

2§. TO‘PLAM TUSHUNCHASI

2.1. To‘plam tushunchasi

To‘plam tushunchasi matematikaning boshlang‘ich tushunchalaridan biri bo‘lib, u misollar yordamida tushuntiriladi. Masalan, shkafdagi kitoblar, barcha to‘g‘ri kasrlar, quyosh sistemasidagi sayyoralar, berilgan nuqtadan o‘tuvchi to‘g‘ri chiziqlar to‘plami haqida gapirish mumkin.

2.2. To‘plam elementi

To‘plamni tashkil etgan narsalar (predmetlar) uning ***elementlari*** deb ataladi.

Odatda, to‘plamlar bosh harflar bilan, uning elementlari esa kichik harflar bilan belgilanadi. Masalan, $A, B, C, \ldots$ larni to‘plam, $a, b, c, \ldots$ larni esa to‘plam elementi deyish mumkin.

Agar A to‘plamning elementi a bo‘lsa, $a\in A$ kabi yoziladi va “a element A to‘plamga tegishli deb o‘qiladi”. Aks holda $a\notin A$ deb yoziladi va “a element A to‘plamga tegishli emas” deb o‘qiladi. Masalan, $A=\{2, 4, 6, 8, 10\}$ bo‘lsa, $6\in A$, $7\notin A$ bo‘ladi.

2.3. Chekli to‘plam

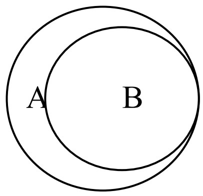
Chekli sondagi elementlardan tashkil topgan to‘plam ***chekli to‘plam*** deb ataladi. Masalan, shkafdagi kitoblar chekli to‘palamni tashkil etadi.

2.4. Cheksiz to‘plam

Matematik analizda ko‘pincha chekli bo‘lmagan to‘plamlarni cheksiz to‘plam deyiladi. Masalan, barcha to‘g‘ri kasrlar, berilgan nuqtadan o‘tuvchi barcha to‘g‘ri chiziqlar to‘plami cheksiz to‘plamlarga misol bo‘ladi.

2.5. Qism to‘plam

Agar B to‘plamning har bir elementi A to‘plamning ham elementi bo‘lsa, B to‘plam A ***to‘plamning qismi*** yoki ***qismiy to‘plami*** (to‘plam osti) deb ataladi va $B \subset A$ kabi belgilanadi (2-rasm). Masalan, $B = \{2,4,6,8\}$, $A = \{1,2,3,4,5,6,7,8\}$ bo‘lsin. Bunda $B \subset A$ ekanligini ko‘rish qiyin emas.



2-rasm

2.6. Bo‘sh to‘plam

Bitta ham elementga ega bo‘lmagan to‘plam ***bo‘sh to‘palam*** deyiladi va $\varnothing$ kabi belgilanadi.

Bo‘sh to‘plam har qanday A to‘plamning qismi (qismiy to‘plami) hisoblanadi.

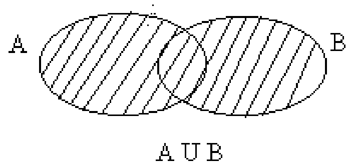
2.7. Teng to‘plamlar

Ta’rif. Agar A va B to‘plamlar uchun $A \subset B$ va $B \subset A$ bo‘lsa, u holda A va B to‘plamlar bir-biriga ***teng to‘plamlar*** deb aytaladi va $A = B$ kabi yoziladi.

Masalan, A to‘plam $k\pi$ ko‘rinishdagi sonlardan iborat bo‘lsin, bunda $k = 0, \pm1, \pm2, \ldots$, ya’ni $A = \{a : a = k\pi, k = 0, \pm1, \pm2, \ldots\}$ va B to‘plam esa $\sin x = 0$ tenglamaning yechimlaridan iborat, ya’ni $B = \{x : \sin x = 0\}$. Agar $\sin x = 0$ tenglamaning barcha yechimlari $x = k\pi$, $k = 0, \pm1, \pm2, \ldots$ formula bilan yozlishini hisobga olsak, $A = B$ bo‘lshini ko‘ramiz.

To‘plamlar ustida amallar

2.8. To‘plamlar yig‘indisi (birlashmasi)



Ta’rif. A va B to‘plamlarning barcha elementlaridan tashkil topgan C to‘plam A va B ***to‘plamlarning yig‘indisi (birlashmasi)*** deyiladi va u$C = A \cup B$ kabi belgilanadi (3-rasm).

3-rasm

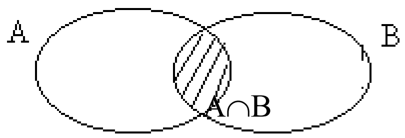
Quyidagilar o‘rinli: $A \cup \varnothing = A,\ A \cup A = A,\ A \cup B = B \cup A$.

Masalan, $A = \{2,4,6,8\}$, $B = \{1,2,3,4\}$ bo‘lsa, unda ularning yig‘indilari quyidagi to‘plamdan iborat bo‘ladi: $A \cup B = \{1,2,3,4,6,8\}$.

Agar $A_1, A_2, ..., A_t$ to‘plamlar berilgan bo‘lsa, ularning yig‘indisi $A_1 \cup A_2 \cup ... \cup A_t$ yuqoridadek ta’riflanadi.

2.9. To‘plamlar ko‘paytmasi (kesishmasi)

Ta’rif. A va B to‘plamlarning barcha umumiy elementlaridan tashkil topgan D to‘plam A va B ***to‘plamlarning ko‘paytmasi (kesishmasi)*** deyiladi. A va B to‘plamlarning ko‘paytmasi $D = A \cup B$ kabi belgilanidi (4-rasm). Masalan,



4-rasm

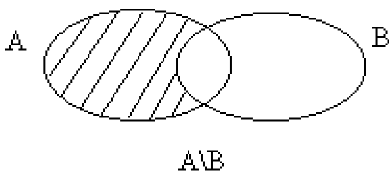
$A = \{2,4,6,8\}$,
$B = \{1,2,3,4\}$ bo‘lsa, ularning ko‘paytmasi $A \cap B = \{2,4\}$ to‘plam bo‘ladi. Ta’rifdan bevosita $A \cap A = A$, $A \cap B = B \cap A$ ekanligi kelib chiqadi, shuningdek, agar $A \subset B$ bo‘lsa, unda $A \cap B = A$ bo‘ladi.

Agar $A_1, A_2, ..., A_t$ to‘plamlar berilgan bo‘lsa, ularning ko‘paytmasi $A_1 \cap A_2 \cap ... \cap A_t$ yuqoridadek ta’riflanadi.

2.10. Kesishmaydigan to‘plamlar

Agar $A \cap B = \varnothing$ bo‘lsa, u holda A va B larga ***kesishmaydigan to‘plamlar*** deyiladi. Masalan, $E = \{2,4,6\}$, $F = \{1,3,5\}$ to‘plamlar kesishmaydigan to‘plamlar bo‘ladi, chunki $E \cap F = \varnothing$.

2.11. To‘plamlar ayirmasi



Ta’rif. A to‘plamning B to‘plamga tegishli bo‘lmagan barcha elementlaridan tuzilgan U to‘plam A to‘plamdan B ***to‘plamning ayirmasi*** deb ataladi (5-rasm).

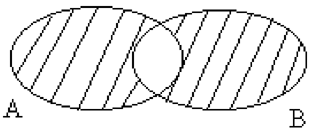
5-rasm

Masalan, agar $A=\{1,2,3,4,5\}$, $B=\{3,6,9,12\}$ bo‘lsa, $A\setminus B=\{1,2,4,5\}$ va $B\setminus A=\{6,9,12\}$ bo‘ladi.

2.12. To‘ldiruvchi to‘plam

Agar A to‘plam S to‘plamning qismi (ya’ni $A\subset S$ bo‘lsa, ushbu $S\setminus A$ ayirma A to‘plamni S to‘plamga ***to‘ldiruvchi to‘plam*** deb ataladi va $C_S A$ kabi yoziladi: $C_S A=S\setminus A$.

2.13. To‘plamlarning simmetrik ayirmasi



Ta’rif. A to‘plamning B to‘plamga tegishli bo‘lmagan barcha elementlaridan va B to‘plamning A to‘plamga tegishli bo‘lmagan barcha elementlaridan tuzilgan to‘plam A va B

6-rasm

to‘plamlarning simmetrik ayirmasi deb ataladi (6-rasm). Simmetrik ayirma $A\Delta B$ kabi belgilanadi.

Ta’rifga ko‘ra

$$A\Delta B=(A\setminus B)\cup(B\setminus A).$$

Masalan, agar $A=\{1,2,3,4,5,6\}$, $B=\{4,5,6,8,7,8,9\}$ bo‘lsa, bu to‘plamning simmetrik ayirmasi $A\Delta B=\{1,2,3,7,8,9\}$ bo‘ladi.

2.14. To‘plamlarning Dekart ko‘paytmasi

Ikki A va B to‘plamlar berilgan bo‘lsin. Birinchi elementi A to‘plamga, ikkinchi elementi B to‘plamga tegishli bo‘lgan tartiblangan (a,b) juftliklarni qaraylik: $a\in A, b\in B$.

Ta’rif. Barcha (a,b) ko‘rinishdagi juftliklardan tuzilgan to‘plam A va B to‘plamlarning ***Dekart ko‘paytmasi*** deb ataladi. To‘plamlarning Dekart ko‘paytmasi $A\times B$ kabi belgilanadi. Odatda $A\times A$ to‘plam A^2 deb belgilanadi.: $A\times A=A^2$. Bunda (a,b) va (b,a) juftliklar $A\times B$ to‘plamning turli elementlari hisoblanadi.

Masalan, $A=\{1,2\}$, $B=\{2,3\}$ to‘plamlar uchun $A\times B=\{\{1,2\},\{1,3\},\{2,2\},\{2,3\}\}$, $B\times A=\{\{2,1\},\{2,2\},\{3,1\},\{3,2\}\}$ bo‘ladi.

Yuqorida to‘plamlar va ular ustida bajarilgan amallarni tasvirlash uchun ishlatilgan shakllar ***Eyler –Venn*** diagrammalari deb ataladi.

2.15. Universal to‘plam

Yuqorida kiritilgan amallar ixtiyoriy to‘plamlar uchun, to‘plamlarning tabiatiga hech qanday shart qo‘ymasdan ta’riflandi. Ma`nosizlik hollarni istisno qilish uchun odatda barcha amallar biror

universal to'plam deb ataluvchi to'plamning qismiy to'plamlari ustida bajariladi deb hisoblanadi va u universal to'plam U yoki Ω bilan belgilanadi.

Matematik analiz kursi davomida universal to'plam sifatida asosan haqiqiy sonlar to'plami R qaraladi.

2.16. To'plamni bo'laklash

Biror A to'plam berilgan bo'lib, $A_1, A_2, ..., A_n$ to'plamlar uning qismiy to'plamlari bo'lsin. $A_k \subset A$ $(k = 1, 2, \ldots, n)$.

Agar $\{A_1, A_2, ..., A_n\}$ qismiy to'plamlar sistemasi uchun:

1) $A_1 \cup A_2 \cup ... \cup A_n = A,$

2) $A_k \cap A_i = \varnothing \quad (k \neq i, k, i = 1, 2, ..., n)$

shartlar bajarilsa, $\{A_1, A_2, ..., A_n\}$ sistema A da bo'laklash bajargan yoki A to'plam $A_1, A_2, ..., A_n$ to'plamlarga ***bo'laklangan*** deyiladi. Ikkala shart birgalikda A dagi har bir element bo'laklashning bitta va faqat bitta elementga tegishli bo'lishini ta'minlaydi.

Masalan, $A = \{1, 2, 3, 4, 5, 6\}$ to'plam berilgan bo'lib, $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, $A_3 = \{6\}$ bo'lsa, 1) $A_1 \cup A_2 \cup A_3 = A$; 2) $A_1 \cap A_2 = \varnothing$, $A_2 \cap A_3 = \varnothing$, $A_1 \cap A_3 = \varnothing$. Demak A to'plam A_1, A_2 va A_3 to'plamlarga bo'lingan.

2.17. To'plamlarni taqqoslash

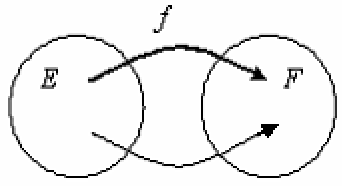
Turli to'plamlarni taqqoslashda, ularni elementlarning miqdori bo'yicha solishtiriladi.

Agar A va B lar chekli to'plamlar bo'lsa, u holda ularning elementlarini bevosita sanash bilan elementlar soni bir-biriga tengligini yoki A to'plamning elementlari soni B to'plamning elementlari sonidan ko'p yoki kam ekanini aniqlash mumkin.

Agar A va B to'plamlar cheksiz to'plamlar bo'lsa, unda bu to'plamlarning elementlarini, ravshanki, sanash yo'li bilan taqqoslab bo'lmaydi. Ammo, bu to'plamlarni ularning elementlarini bir-biriga mos qo'yish yo'li bilan taqqoslash mumkin.

2.18. Aksalantirishlar

Ta'rif. Agar E to'plamdan olingan har bir elementga ($x \in E$) biror qonun yoki qoidaga ko'ra F to'plamda bitta y element ($y \in F$) mos qo'yilgan bo'lsa, u holda E to'plam F to'plamga ***akslantirilgan*** deyiladi va $f: E \to F$ yoki $x \xrightarrow{f} y$ kabi belgilanadi.



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Bunda E to'plam f akslantirishning ***aniqlanish sohasi*** deyiladi.

2.19. Ichiga akslantirish

Ta'rif. Agar $f: E \to F$ akslantirishda $f(E) \neq F$ bo'lsa, bu akslantirishga E to'plamni F ning ***ichiga akslantirish*** deyiladi.

2.20. Ustiga (syuryektiv) akslantirish

Ta'rif. Agar $f: E \to F$ akslantirishda $f(E) = F$ bo'lsa, bu akslantirishga E to'plamni F ning ***ustiga (syuryektiv) akslantirish*** deyiladi.

2.21. O'zaro bir qiymatli (biyektiv) akslantirish (moslik)

Agar $f: E \to F$ akslantirish ustiga akslantirish bo'lsa va ixtiyoriy $y \in F$ element yagona elementning aksi bo'lsa, f akslantirish ***o'zaro bir qiymati akslantirish*** deyiladi.

2.22. Ekvivalent to'plamlar

Ta'rif. Agar A va B to'plamlar elementlari orasida o'zaro bir qiymatli moslik o'rnatish mumkin bo'lsa, ular bir-biriga ***ekvivalent to'plamlar*** deb ataladi.

Ekvivalent A va B to'plamlar $A \sim B$ kabi belgilanadi. Masalan, $A = \{1, 2, 3, 4, 5\}$, $B = \{10, 11, 12, 13, 14\}$ bo'lsa, $A \sim B$.

2.23. Sanoqli to'plam

Ta'rif. Natural sonlar to'plami N ga ekvivalent bo'lgan har qanday *to'plam* ***sanoqli to'plam*** deb ataladi.

Natural sonlar to'plami N ga ekvivalent bo'lgan barcha to'plamlar ***sanoqli to'plamlar sinfini*** tashkil etadi.

To'plamning qismi o'ziga ekvivalent bo'lishi faqat cheksiz to'plamlargagina xosdir.

2.24. To‘plam quvvati

Ekvivalent to‘plamlar sinfining miqdoriy xarakteristikasi sifatida to‘plamning quvvati tushunchasi kiritiladi. Chekli to‘plamlar uchun quvvat to‘plam elementlarining sonidan iborat.

2.25. O‘rganish usullari

Odatda jarayonlar ***deduktiv*** va ***induktiv*** fikrlashlar (usullar) yordamida o‘rganiladi.

Deduktiv usulda o‘rganishda umumiy tasdiqlardan xususiy tasdiqlarni o‘rinli ekanigini mantiqan fikrlab ko‘rsatiladi.

Induktiv usulda esa tasdiq bir nechta to‘g‘ri oddiy xususiy hollardan umumiy hol uchun to‘g‘riligi ko‘rsatiladi.

2.26. Matematik induksiya usuli

Biror n natural songa bog‘liq $A(n)$ gipoteza bayon qilingan bo‘lsin. Matematik induksiya yordamida gipoteza quyidagicha o‘rganiladi:

1. $n=1$ da mulohazaning to‘g‘riligi tekshiriladi;
2. $n=k$ da $A(k)$ mulohaza to‘g‘ri deb qabul qilinadi;
3. $n=k+1$ da $A(k)$ to‘g‘riligidan foydalanib $A(k+1)$ uchun to‘g‘riligi ko‘rsatiladi.

Shundan so‘ng, $A(n)$ mulohaza barcha n lar uchun o‘rinli deb xulosa qilinadi.

Misol. Quyidagi formulani

$$1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

ixtiyoriy $n\in N$ da o‘rinli ekanligini isbot qiling.

1. $n=1$ da tenglikning ikkala tomoni 1 ga teng, ya’ni

$$1^3=\left(\frac{1(1+1)}{2}\right)^2=1^2$$

bo‘lib, matematik induksiyaning birinchi sharti bajariladi.

2. Faraz qilamiz $n=k$ da formula o‘rinli, ya’ni

$$1^3+2^3+3^3+\ldots+k^3=\left(\frac{k(k+1)}{2}\right)^2$$

bo‘lsin. Bu tenglikni ikkala tomoniga $(k+1)^3$ hadni qo‘shamiz

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3=(k+1)^2\left(\frac{k^2}{4}+k+1\right)=$$

$$=(k+1)^2\left(\frac{k^2+4k+4}{4}\right)=(k+1)^2\cdot\left(\frac{k+2}{2}\right)^2=\left(\frac{(k+1)(k+2)}{2}\right)^2.$$

Shunday qilib formulaning $n=k$ to‘g‘riligidan uning $n=k+1$ uchun o‘rinliligi kelib chiqdi. Bu esa, formulaning barcha n lar uchun o‘rinli bo‘lishini bildiradi.

3§. HAQIQIY SONLAR

3.1. Natural sonlar. Tartiblangan to‘plam

Sanashda ishlailadigan sonlar natural sonlar to‘plamini tashkil qiladi. Barcha natural sonlar $\{1,\ 2, 3, ...\}$ to‘plami N harfi bilan belgilanadi. Bu to‘plamdan olingan ixtiyoriy natural son n, m va p lar uchun quyidagi ikki tasdiq doimo o‘rinli:

1) $n=m, n>m, n<m$ munosabatlardan bittasi va faqat bittasi o‘rinli;

2) $n<m, m<p$ tengsizliklardan $n<p$ tengsizlik o‘rinli ekanligi kelib chiqadi.

Agar biror E to‘plamning elementlari uchun yuqorida keltirilgan 1) va 2) tasdiqlar o‘rinli bo‘lsa, E to‘plam ***tartiblangan to‘plam*** deyiladi. Natural sonlar to‘plami tartiblangan to‘plam.

Natural sonlar to‘plami quyidan 1 bilan chegaralangan, yuqoridan chegaralanmagan.

3.2. Natural sonlar ustida bajariladigan amallar

Natural sonlar to‘plami N da ikkita amal qo‘shish $(n+m)$ va ko‘paytirish $(n\cdot m)$ amallari doimo bajariladi va ular quyidagi xossalarga ega.

1°. Kommutativlik: $n+m=m+n, \quad n\cdot m=m\cdot n$.

2°. Assotsiativlik: $(n+m)+p=n+(m+p), \quad (n\cdot m)\cdot p=n\cdot(m\cdot p)$.

3°. Distributivlik: $(n+m)\cdot p=n\cdot p+m\cdot p$.

4°. N to‘plamda shunday k element borki, $k\cdot n=n\cdot k=n$ bo‘ladi. Bu element $k=1$ dir.

3.3. Butun sonlar

Natural sonlarga qarama-qarshi sonlar manfiy butun sonlar (manfiy natural) deyiladi.

Barcha manfiy natural sonlar, nol soni va barcha natural sonlar ***butun sonlar*** deyiladi va u Z harfi bilan belgilanadi:

$$Z=\{...,-4,-3,-2,-1,0,1,2,3,4,...\}.$$

Bu to'plamdagi 0 soni ixtiyoriy $n\in N$ uchun $n+0=n$ va $n+(-n)=0$ tengliklarni qanoatlantiradi. Bu yerda n va $-n$ o'zaro ***qarama-qarshi sonlar*** deb ataladi. Ravshanki, $N\subset Z$.

Butun sonlar to'plami tartiblangan. Butun sonlar to'plami ham quyidan, ham yuqoridan chegaralanmagan.

3.4. Butun sonlar to'plamida bajariladigan amallar

1°. Kommutativlik: $n+m=m+n, \quad n\cdot m=m\cdot n$.

2°. Assotsiativlik: $(n+m)+p=n+(m+p), \quad (n\cdot m)\cdot p=n\cdot(m\cdot p)$.

3°. Distributivlik: $(n+m)\cdot p=n\cdot p+m\cdot p$.

4°. Z to'plamda shunday k element borki, $k\cdot n=n\cdot k=n$ bo'ladi. Bu element $k=1$ dir.

5°. Ixtiyoriy $q\in Z$ element uchun Z to'plamda shunday element $-q$ mavjudki, $q+(-q)=0$ bo'ladi.

6°. Ixtiyoriy $q\in Z$ element uchun $q+0=0+q=q$ bo'ladi.

7°. Ixtiyoriy $q\in Z$ element uchun $q\cdot 0=0\cdot q=0$ bo'ladi.

3.5. Ratsional sonlar

Ushbu qisqarmaydigan $r=\dfrac{p}{n};\ p\in Z,\ n\in N$ kasr ko'rinishida tasvirlanadigan har bir son ***ratsional son*** deyiladi. Ratsional sonlar to'plami Q harfi bilan belgilanadi:

$$Q=\{r: r=\frac{p}{n},\ p\in Z,\ n\in N,\quad EKUB(p,n)=1\}.$$

Ravshanki, $N\subset Z\subset Q$.

Ratsional sonlar to'plami tartiblangan. Ratsional sonlar to'plami ham quyidan, ham yuqoridan chegaralanmagan.

3.6. Ratsional sonlar to'plamida bajariladigan amallar

Ratsional sonlar to'plamida qo'shish, ko'paytirish, ayirish amallari bilan bir qatorda bo'lish amali (nolga teng bo'lmagan songa) ham kiritiladi va bu amallarga nisbatan ushbu xossalar o'rinli (bu xossalarda r,t va s lar ixtiyoriy ratsional sonlar):

1°. Kommutativlik: $r+t=t+r, \quad rt=tr$.

2°. Assotsiativlik: $(r+t)+s=r+(t+s), \quad (r\cdot t)\cdot s=r\cdot(t\cdot s)$.

3°. Distributivlik: $(r+t)\cdot s=r\cdot s+t\cdot s$.

4°. Nol sonining xususiyati: $r+0=r, \quad r\cdot 0=0$.

5°. Bir sonining xususiyati: $r\cdot 1=r$.

6°. Qarama-qarshi elementning mavjudligi: $\forall r\in Q$ uchun shunday $-r\in Q$ son mavjudki, $r+(-r)=0$ bo‘ladi.

7°. Teskari elementning mavjudligi: $\forall r\in Q$ $(r\neq 0)$ uchun shunday $r^{-1}\in Q$ son mavjudki, $r\cdot r^{-1}=1$ bo‘ladi.

8°. Ixtiyoriy $r\in Q$, $t\in Q$, $s\in Q$ sonlar uchun $r>t$ bo‘lganda $r+s>t+s$.

9°. Ixtiyoriy $r\in Q$, $t\in Q$, $s\in Q$ $(s>0)$ sonlar uchun $r>t$ bo‘lganda $r\cdot s>t\cdot s$ bo‘ladi.

10°. Ixtiyoriy ikki musbat r va t ratsional sonlar uchun shunday natural son n mavjudki, $n\cdot r>t$ bo‘ladi. Bu xossa odatda ***Arximed aksiomasi*** deyiladi.

3.7. Ratsional sonlar to‘plamining asosiy xossalari

Ratsional sonlar to‘plami ikkita asosiy xossaga ega: tartiblanganlik va zichlik.

Ratsional sonlar to‘plamining tartiblanganligi. Ratsional sonlar to‘plami Q dan olingan ixtiyoriy ratsional r,s,t sonlar uchun quyidagi ikki tasdiq o‘rinli bo‘ladi:

1) $r=s, r>s, r<s$ munosabatlardan bittasi va faqat bittasi o‘rinli.

2) $r<s, s<t$ tengsizliklardan $r<t$ tengsizlikning o‘rinli bo‘lishi kelib chiqadi.

Bu hol ratsional sonlar to‘plami Q ning tartiblanganlik xossasini ifodalaydi.

Ratsional sonlar to‘plamining zichligi. Faraz qilaylik, $r\in Q, t\in Q$ va $r<t$ bo‘lsin. U holda $\frac{r+t}{2}\in Q$ va $r<\frac{r+t}{2}<t$ bo‘ladi. Bu esa r va t ratsional sonlar orasida $\frac{r+t}{2}$ ratsional son bor ekanligini ko‘rsatadi. $\frac{r+t}{2}$ sonni s bilan belgilab, r va s sonlari orasida joylashgan $\frac{r+s}{2}$ hamda s va t orasida joylashgan $\frac{s+t}{2}$ ratsional sonlar borligini ko‘ramiz: $r<\frac{r+s}{2}<\frac{r+t}{2}<\frac{s+t}{2}<t$.

Bu jarayonni istagancha davom ettirish yo‘li bilan ixtiyoriy r va t ratsional sonlar orasida cheksiz ko‘p ratsional sonlar borligi aniqlanadi. Mana shu xossa ratsional sonlar to‘plami Q ning ***zichligi xossasi*** deyiladi.

3.8. To‘plamning eng katta elementi

Ta’rif. Agar shunday $r^* \in A$ $(A \subset Q)$ topilib, $\forall r \in A$ uchun $r \le r^*$ tengsizlik bajarilsa, r^* ratsional son A ***to‘plamning eng katta elementi*** deyiladi.

3.9. To‘plamning eng kichik elementi

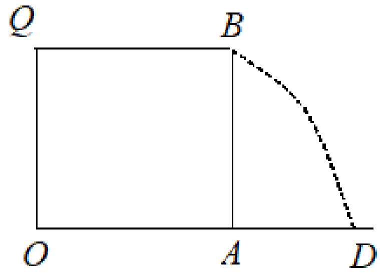
Ta’rif. Agar shunday $r_* \in A$ $(A \subset Q)$ topilib, $\forall r \in A$ uchun $r \ge r_*$ tengsizlik bajarilsa, r_* ratsional son A ***to‘plamning eng kichik elementi*** deyiladi.

3.10. Ratsional sonlar to‘plamini kengaytirish zaruriyati

Tomoni bir birlikka teng bo‘lgan $QABC$ kvadratni qaraylik.

Bu kvadratnng diagonali OB ning uzunligi $\sqrt{2}$ ga teng. Sirkulning uchini 0 nuqtada qo‘yib, radiusi OB ga teng bo‘lgan aylana chizaylik. Bu aylana OA tomon joylashgan to‘g‘ri chiziqni D nuqtada kesadi.

$OA < OB$ bo‘lgani uchun D nuqta A nuqtadan o‘ngga joylashgan bo‘ladi. Ravshanki, $OB = OD = \sqrt{2}$. Demak, D nuqtaga $\sqrt{2}$ son mos keladi. $\sqrt{2}$ esa ratsional son emas.



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Bu quyidagi teoremada isbotlanadi.

Teorema. Ratsional sonlar to‘plami Q da kvadratni 2 ga teng bo‘lgan ratsional son mavjud emas.

Teskarisini faraz qilaylik, ya’ni Q to‘plamda shunday qisqarmaydigan $\frac{p}{n}$ $(p \in Z, n \in N)$ kasr ko‘rinishida yoziladigan ratsional son borki,

$$\left(\frac{p}{n}\right)^2 = 2 \qquad (1)$$

tenglik o‘rinli bo‘lsin. Yuqoridagi tenglikni quyidagicha

$$p^2 = 2n^2 \qquad (2)$$

yozib olamiz. Bundan p juft son ekanligi ko‘rinadi. Demak $p = 2m$ $(m \in Z)$. Buni (2) ga qo‘ysak $n^2 = 2m^2$ hosil bo‘ladi. Bu esa n sonning ham juft ekanligini ko‘rsatadi. Demak, yuqoridagi farazdan p va n sonlarning juft sonligi kelib chiqadi. Binobarin ular uchun 2 umumiy ko‘paytuvchi. Bu esa $\frac{p}{n}$ sonning qisqarmaydigan kasr ekaniga zid.

Shunday qilib, to‘g‘ri chiziqda olingan har bir nuqtada Q to‘plamda unga mos keladigan ratsional son mavjud bo‘lavermas ekan.

Ratsionl sonlar to‘plamini kengaytirishda bir-biriga ekvivalent bo‘lgan bir necha usullar mavjud: Koshi usuli, Kantor usuli, Veyershtrass usuli hamda Dedekind usuli.

3.11. Ratsional sonlar to‘plamida kesim

Ta’rif. Ratsional sonlar to‘plami Q shunday A va A' to‘plamlarga ajratilib, bunda

1) $A \neq \varnothing, A' \neq \varnothing$
2) $A \cup A' = Q$
3) $\forall a \in A, \forall a` \in A' \Rightarrow a < a'$

shartlar qanoatlantirilsa, A va A' to‘plamlar Q to‘plamda ***kesim bajaradi*** deyiladi. Bunda A to‘plam ***kesimning quyi sinfi***, A' esa ***yuqori sinfi*** deyiladi. Kesim (A, A') kabi belgilanadi.

Masalan: r_0 va undan kichik bo‘lgan barcha ratsional sonlardan iborat to‘plam A, r_0 dan katta bo‘lgan barcha ratsional sonlar to‘plami A' bo‘lsin: $A = \{r : r \in Q, r \leq r_0\}, A' = \{r : r \in Q\ r_0 > r\}$. Bu A va A' to‘plamlar uchun ta’rifidagi uchala shart o‘rinli bo‘ladi. Demak, bunday tuzilgan A va A' to‘plamlar Q da kesim bajaradi. Odatda bu kesim $r_0 = (A, A')$ kabi belgilanadi.

<u>I tur kesim</u>. Kesimning quyi sinfi A da eng katta element (r_0 ratsional son) mavjud, kesimning yuqori sinfi A'da esa eng kichik element mavjud emas. Bunda r_0 ratsional son quyi sinf A ning yopuvchi elementi bo‘ladi.

<u>II tur kesim</u>. Kesimning quyi sinfi A ga eng katta element mavjud emas, kesimning yuqori sinfi A' da esa kichik element (r_0', p ratsional son) mavjud. Bunda r_0' ratsional son yuqori sinf A' ning yopuvchi element bo‘ladi.

<u>III tur kesim</u>. Kesimning quyi sinfi A da eng katta element va kesimning yuqori sinfi A' da eng kichik element mavjud emas. Bunda quyi sinf A da, yuqori sinf A' da yopuvchi elementlar mavjud emas.

Birinchi va ikkinchi tur kesimlarda ularning quyi yoki yuqori sinflarni yopiq bo‘lib, yopuvchi elementlarni bir sinfdan ikkinchi sinfga o‘tkazib, har doim bir turdagi kesimga – quyi sinfi ochiq, yuqori sinfi esa yopiq bo‘lgan kesimga keltirish mumkin. Quyi sinfda eng katta element mavjud bo‘lmagan (ochiq sinf) , yuqori sinfda esa eng kichik element mavjud bo‘lgan (yopiq sinf) kesim ***ratsional kesim*** deyiladi.

Q da bajariladigan har bir ratsional kesim bitta ratsional sonni aniqlaydi.

3.12. Irratsional son

Irratsional sonlar deb $\frac{m}{n}, \ (m \in Z, \ n \in N)$ ko‘rinishida yozib bo‘lmaydigan sonlarga aytiladi. Masalan, $\pi, \sqrt{2}, \sqrt{5}, \ldots$. Bunday sonlar to‘plami I yoki U harflari bilan belgilanadi.

3.13. Irratsional kesim

Ratsional sonlar to‘plami Q da bajarilgan III tur kesim quyi sinfi ham, yuqori sinfi ham ochiq bo‘lgan kesim ***irratsional kesim*** deyiladi.

Ta’rif. Ratsional sonlar to‘plami Q da bajarilgan kesim ***irratsional sonni*** aniqlaydi deyiladi.

3.14. Haqiqiy sonlar

Ta’rif. Ratsional hamda irratsional sonlar umumiy nom bilan ***haqiqiy sonlar*** deyiladi.

Barcha haqiqiy sonlar to‘plami R harfi bilan belgilanadi. Tarifga ko‘ra, $R = Q \cup U$. R-lotincha realus – “haqiqiy” degan ma`noni anglatuvchi so‘zning bosh harfi.

3.15. Haqiqiy sonlar to‘plamida kesim

Ta’rif. Haqiqiy sonlar to‘plami R shunday E va E' to‘plamlarga ajratilib, unda

1) $E \neq \varnothing, \ E' \neq \varnothing$
2) $E \cup E' = R$
3) $\forall x \in E, \forall x' \in E' \Rightarrow x < x'$

shartlar bajarilsa, E va E' to‘plamlar R to‘plamda ***kesim bajaradi*** deyiladi va (E, E') kabi belgilanadi.

Bunda E to‘plam kesimning quyi sinfi, E' to‘plam esa kesimning yuqori sinfi deyiladi.

Masalan, ushbu $E = \{x : x \in R, x \leq x_0\}, \quad E^{`} = \{x : x \in R, x > x_0\} \quad (x_0 \in R)$ to‘plamlar R da (E, E') kesim bajaradi. Bu kesimning quyi sinfi E da eng katta element (u x_0 ga teng) bo‘lib, yuqori sinfi E da eng kichik element bo‘lmaydi.

3.16. Dedekind teoremasi

Teorema. Haqiqiy sonlar to‘plami R da bajarilgan har qanday (E, E') kesim uchun faqat quyidagi ikki holdan biri bo‘lishi mumkin:

a) Kesimning quyi sinfi – E da eng katta element mavjud, yuqori sinf – E' da esa eng kichik element mavjud emas;

b) kesimning quyi sinfi – E da eng katta element mavjud emas, yuqori sinf – E' da esa eng kichik element mavjud.

Teorema. R da bajarilgan har qanday (E, E') kesim yagona haqiqiy sonni aniqlaydi.

3.17. Haqiqiy sonlarning asosiy xossalari

Haqiqiy sonlar to‘plami tartiblanganlik va zichlik xossalariga ega.

I. **Tartiblanganlik xossasi**.

1) Ixtiyoriy ikki x va y sonlar berilgan bo‘lsa, unda

$$x = y, \quad x < y, \quad x > y$$

munosabatlardan bittasi va faqat bittasi o‘rinli bo‘ladi;

2) x, y, z haqiqiy sonlar uchun ushbu $x < y$, $y < z$ tengsizliklardan $x < z$ tengsizlik kelib chiqadi.

II. **Zichlik (uzluksizlik) xossasi**. Ixtiyoriy ikkita x va y haqiqiy sonlar bo‘lib, $x < y$ ($x > y$) bo‘lsa, u holda shunday r ratsional son mavjudki, $x < r < y$ ($x > r > z$) tengsizlik o‘rinli bo‘ladi. Demak, ixtiyoriy ikkita bir-biriga teng bo‘lmagan haqiqiy sonlar orasida kamida bitta haqiqiy son mavjud. Bundan esa ular orasida cheksiz ko‘p haqiqiy son mavjudligi kelib chiqadi. Demak, R-zich to‘plam ekan.

3.18. Haqiqiy sonlarning asosiy aksiomalari

I. **Qo‘shish amali**. Ixtiyoriy tartiblangan bir juft a va b haqiqiy sonlar uchun yagona usul bilan aniqlangan va ularning ***yig‘indisi*** deb ataluvchi $a+b$ ko‘rinishda belgilanuvchi son mavjud. Quyidagi xossalar o‘rinli:

I$_1$. Kommutativlik: $a+b=b+a$.

I$_2$. Assotsiativlik: $a+(b+c)=(a+b)+c$.

I$_3$. Ixtiyoriy $a \in R$ uchun $a+0=a$.

I$_4$. Ixtiyoriy $a \in R$ uchun va unga qarama-qarshi $-a$ son uchun

$$a+(-a)=0.$$

II. <u>**Ko‘paytirish amali**</u>. Ixtiyoriy tartiblangan bir juft a va b haqiqiy sonlar uchun yagona usul bilan aniqlangan va ularning ***ko‘paytmasi*** deb ataluvchi $a\cdot b$ ko‘rinishda belgilanuvchi son mavjud. Quyidagi xossalar o‘rinli:

II₁. Kommutativlik: $a\cdot b=b\cdot a$.

II₂. Assotsiativlik: $a\cdot(bc)=(ab)\cdot c$.

II₃. Bir sonining xususiyati: $a\cdot 1=a$.

II₄. Teskari sonning mavjudligi. $\forall a\in R$ ($a\neq 0$) ga teskari son deb ataluvchi va $\frac{1}{a}$ ko‘rinishida belgilanuvchi son mavjud bo‘lib

$$a\cdot\frac{1}{a}=1.$$

III. Qo‘shish va ko‘paytirish amallari orasidagi bog‘liqlik. Distributivlik (***qo‘shishning ko‘paytirishga nisbatan taqsimot qonuni***)

$$(a+b)c=ac+bc.$$

IV. <u>**Tartiblanganlik.**</u> Ixtiyoriy a haqiqiy son uchun quyidagilarning biri o‘rinli: $a>0$ (a noldan katta), $a=0$ (a nolga teng) yoki $a<0$ (a kichik noldan), shu bilan birga $a>0$ shart $-a<0$ shart bilan teng kuchli.

Agar $a>0$, $b>0$ bo‘lsa quyidagilar o‘rinli:

IV₁. $a+b>0$.

IV₂. $ab>0$.

V. <u>**Uzluksizlik**</u>. Ixtiyoriy $a\in A\subset B$, $b\in B\subset R$ va $a\leq b$ tengsizlik o‘rinli bo‘lganda, shunday α son mavjudki

$$a\leq\alpha\leq b$$

tengsizlik o‘rinli bo‘ladi.

Elementlari I-V xossalarga ega bo‘lgan to‘plam haqiqiy sonlar to‘plami deyiladi. Bu to‘plamning har bir elementi ***haqiqiy son*** deyiladi.

3.19. Qo‘shish va ko‘paytirishning asosiy xossalari

<u>**Ayirish amali.**</u> Qo‘shishga nisbatan teskari amal bu ayirish amali.

Ixtiyoriy tartiblangan a va b haqiqiy sonlar uchun $a+(-b)$ son a va b sonlar ayirmasi deyiladi va $a-b$ kabi belgilanadi, ya’ni

$$a-b=a+(-b).$$

Agar

$$a+b=c$$

bo'lsa, tenglikni ikkala tomoniga $-b$ ni qo'shib $(a+b)+(-b)=c+(-b)$ tengsizlikni hosil qilamiz. I_2 dan va ayirish ta'rifidan

$$a+(b+(-b))=c-b,$$

hamda $b+(-b)=0$ dan

$$a=c-b.$$

1°. Nol xususiyatga ega bo'lgan son yagonadir. Faraz qilaylik nol xususiyatga ega bo'lgan 0 va $0'$ ikkita nol bo'lsin. U holda I_3 dan $0'+0=0'$, $0+0'=0$. I_2 dan tenglikning o'ng tomonlari teng, bundan chap tomonlar ham teng bo'lishi kerak: $0=0'$.

2°. Berilgan songa qarama-qarshi son yagonadir. b va c sonlar a songa qarama-qarshi sonlar bo'lsin, ya'ni $a+b=0$ va $a+c=0$. Unda birinchi tenglikdan $a+b+c=0+c$ yoki $(a+b)+c=c$ bundan $(a+c)+b=c$, lekin $a+c=0$, natijada $b=c$.

3°. Ixtiyoriy a son uchun

$$-(-a)=a$$

tenglik o'rinli.

Haqiqatan, $a+(-a)=0$ tenglik kommutativligidan $-a+a=0$. Bundan $a=-(-a)$.

4°. Ixtiyoriy a son uchun

$$a-a=0$$

tenglik o'rinli. Haqiqatdan,

$$a-a=a+(-a)=0.$$

5°. Ixtiyoriy a va b sonlar uchun

$$-a-b=-(a+b)$$

tenglik o'rinli. Haqiqatdan,

$$a+b+(-a-b)=(a-a)+(b-b)=0.$$

6°. $a+x=b$ tenglama R da yagona

$$x=b-a$$

ildizga ega. Haqiqatdan, agar $a+b=c$ bo'lsa, $a=c-b$ ni hisobga olsak, $x=b-a$. Ildiz mavjudligini ko'rsatish uchun, $x=b-a$ son ildiz ekanligini tekshirish kerak:

$$a+(b-a)=a+[b+(-a)]=[a+(-a)]+b=b.$$

Bo'lish amali. Ko'paytirish amaliga teskari amal, bu bo'lish amalidir.

Ta'rif. Ixtiyoriy tartiblangan a va b $(b \neq 0)$ sonlar uchun, $a \cdot \frac{1}{b}$ son a sonni b songa bo'lish deyiladi va $\frac{a}{b}$ yoki $a:b$ kabi belgilanadi.

7°. Bir soni xususiyatiga ega bo'lgan son yagona.
8°. Noldan farqli berilgan songa teskari son yagonadir.
9°. Ixtiyoriy $a \neq 0$ son uchun quyidagi tenglik o'rinli

$$\frac{1}{\frac{1}{a}} = a.$$

10°. Ixtiyoriy $a \neq 0$ son uchun

$$\frac{a}{a} = 1.$$

11°. Ixtiyoriy $a \neq 0$, $b \neq 0$ uchun

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

12°. $ax = b$ tenglama $a \neq 0$ da haqiqiy sonlar to'plamida $x = \frac{b}{a}$ yagona yechimga ega.

7°-12° xossalar xuddi 1°-6° xossalar kabi isbot qilinadi.

13°. $\frac{a}{b} = \frac{c}{d}$ tenglik $b \neq 0$, $d \neq 0$ da faqat $ad = bc$ shart bajarilganda o'rinli.

14°. Ixtiyoriy a, b va c sonlar uchun

$$a(b-c) = ab - ac$$

tenglik o'rinli.

Haqiqatdan,

$$a(b-c) = a(b-c) + ac - ac = a(b-c+c) - ac = ab - ac.$$

15°. Ixtiyoriy a son uchun

$$a \cdot 0 = 0.$$

Ixtiyoriy b son uchun $b - b = 0$. 4° dan va 14° dan

$$a \cdot 0 = a(b-b) = ab - ab = 0$$

ekanligi kelib chiqadi.

16°. Agar $ab = 0$ bo'lsa, ko'paytuvchilardan hech bo'lmaganda biri nolga teng.

Faraz qilaylik $a \neq 0$, $ab = 0$ ni $\frac{1}{a}$ ga ko'paytiramiz. $\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$, bundan $\left(\frac{1}{a} \cdot a\right) b = 0$ bundan $b = 0$.

17°. Ixtiyoriy a va b sonlar uchun

$$(-a)b = -ab, \quad (-a)(-b) = ab$$

bo‘ladi va xususiy holda $(-1)a = -a$.

Haqiqatdan,

$$(-a)b = (-a)b + ab - ab = (-a + a)b - ab = -ab.$$

Bundan,

$$(-a)(-b) = -a(-b) = (-1)[a(-b)] = (-1)(-ab) = -(-ab) = ab.$$

18°. Kasrlarni qo‘shish

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad (b \neq 0, \ d \neq 0)$$

qoidasi bilan bajariladi.

Isboti. Yuqorida keltirilgan qoidalardan foydalanamiz:

$$\frac{ad + bc}{bd} = (ad + bc)\frac{1}{bd} = ad\frac{1}{bd} + bc\frac{1}{bd} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{a}{b} + \frac{c}{d}.$$

19°. Kasrlarni ko‘paytirish

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad (b \neq 0, \ d \neq 0)$$

qoida bilan bajariladi.

$$11^\circ \Rightarrow \frac{ac}{bd} = ac \cdot \frac{1}{bd} = ac\frac{1}{b}\frac{1}{d} = \left(a \cdot \frac{1}{b}\right) \cdot \left(c \cdot \frac{1}{d}\right) = \frac{ac}{bd}.$$

20°. $\frac{a}{b}$ $(a \neq 0, \ b \neq 0)$ kasrga teskari kasr $\frac{b}{a}$ bo‘lib, $\frac{a}{b} \cdot \frac{b}{a} = 1$ (kasrlarni ko‘paytirish xossasidan kelib chiqadi).

21°. Kasrlarni bo‘lish

$$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc} \quad (b \neq 0, c \neq 0, d \neq 0)$$

qoida bilan bajariladi.

<u>Daraja.</u> a haqiqiy son va n natural son berilgan bo‘lsin.

Ta’rif. a sonining n marta o‘zini o‘ziga ko‘paytmasi a sonining n***-darajasi*** deyiladi va a^n kabi belgilanadi. Demak,

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \ marta}.$$

Ta’rifdan $a^0 = 1$ va ixtiyoriy $n \in N$ uchun

$$a^{-n} = \frac{1}{a^n}.$$

22°. Agar m va n butun sonlar bo‘lsa ($a \neq 0$),

$$a^m a^n = a^{m+n}, \quad (a^m)^n = a^{mn}$$

tengliklar o‘rinli.

Agar $m = 0, n = 0$ bo‘lsa, tenglik o‘rinliligi aniq. Agar m, n natural sonlar bo‘lsa, daraja ta’rifidan

$$a^m\cdot a^n=\underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{m\ marta}\cdot\underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{n\ marta}=a^{m+n}.$$

Agar $m<0, n>0$ va $a\neq 0$ da $k=-m$ hamda $k\leq n$ da

$$a^m\cdot a^n=a^{-k}a^n=\frac{a^n}{a^k}=\frac{\overbrace{a\cdot a\cdot a\cdot\ldots\cdot a}^{n\ marta}}{\underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{k\ marta}}=a^{n-k}=a^{m+n}$$

va $k>n$ da

$$a^m a^n=\frac{a^n}{b^k}=\frac{1}{a^{k-n}}=a^{n-k}=a^{m+k}.$$

Agar $m<0, n<0$ va $a\neq 0$ bo‘lsa $k=-m$, $l=-n$ deb 11° dan foydalanib

$$a^m a^n=a^{-k}\cdot a^{-l}=\frac{1}{a^k}\cdot\frac{1}{a^l}=\frac{1}{a^{k+l}}=a^{-(k+l)}=a^{m+n}.$$

Elementlari I, II, III shartlarni qanoatlantiruvchi hech bo‘lmaganda bitta elementga ega bo‘lgan to‘plamga ***maydon*** deyiladi.

Ratsional sonlar, haqiqiy sonlar, kompleks sonlar va ratsional funksiyalar maydon hosil qiladi.

3.20. Son qiymatlarini taqqoslash va ularning asosiy xossalari

Ta’rif. Agar $b-a>0$ bo‘lsa, b son a sondan katta deyiladi va $b>a$ ko‘rinishida yoziladi yoki a son b sondan kichik deyiladi va $a<b$ ko‘rinishida yoziladi. Taqqoslashning asosiy xossalarini keltiramiz.

1°. Agar $a>b$ va $b>c$ bo‘lsa, $a>c$ bo‘ladi (tranzitivlik xossasi). Agar $a>b$ va $b>c$ bo‘lsa, ta’rifdan $a-b>0$ va $b-c>0$. IV$_1$ dan

$$(a-b)+(b-c)>0,\quad a+b-b-c=a-c>0.$$

2°. Agar $a>b$ bo‘lsa, ixtiyoriy c sonda $a+c>b+c$ o‘rinli. Oldingi bo‘limdagi 5° dan,

$$a-b=a+c-c+b=(a+c)-(b+c)>0$$

(chunki $a>b$), natijada $a+c>b+c$.

$a<b$ ko‘rinishdagi munosabat “a kichik b” va $a=b$ “a teng b” hamda $a>b$ “a katta b” deb o‘qiladi.

$a\leq b$ yozuv $b\geq a$ yozuviga teng kuchli bo‘lib, $a=b$ yoki $a<b$ ma’noni anglatadi.

3°. Ixtiyoriy ikkita a va b sonlar uchun quyidagi munosabatlardan faqat bittasi o‘rinli bo‘ladi:

$$a>b,\ a=b \text{ yoki } a<b.$$

4°. Agar $a<b$ bo‘lsa, $-a>-b$ o‘rinli.

Haqiqatdan, $a<b$ shartdan $-a=-a+b+(-b)=(b-a)+(-b)>0+(-b)=-b$.

5°. Agar $a<b$ va $c\le d$ bo‘lsa, $a+c<b+d$ o‘rinli, ya’ni bir xil ishorali tengsizliklarni hadlab qo‘shish mumkin.

6°. Agar $a<b$ va $c\ge d$ bo‘lsa, $a-c<b-d$ o‘rinli.

Haqiqatdan, $c\ge d$ va 4° dan $-c\le -d$. $a<b$ va $-c\le -d$ tengsizliklarni qo‘shish,
$a-c<b-d$ ega bo‘lamiz.

7°. Agar $a<b$ va $c<0$ bo‘lsa, $ac>bc$ o‘rinli.

Shu bo‘limdagi 4° dan $-c>0$ va IV$_2$ dan $a(-c)<b(-c)$. Oldingi bo‘limdagi 17° dan, $ac<-cb$. Natijada 4° dan, $ac>bc$.

Natija. Agar $b>0$, $d>0$ bo‘lsa,

$$\frac{a}{b}>\frac{c}{d}$$

shart $ad<bc$ shartga teng kuchli.

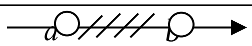
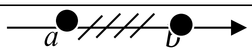
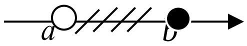
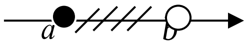
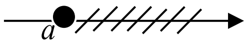
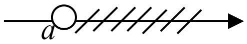
3.21. Sonli oraliqlar (sonli to‘plamlar)

Elementlari haqiqiy sonlardan iborat bo‘lgan to‘plam ***sonli to‘plam*** deyiladi va $E=\{x\}$ kabi belgilanadi.

Matematik analizda asosan sonli to‘plamlar qaraladi.

Sonli oraliqlar turi, geometrik tasviri, belgilanishi va tengsizliklar yordamida ifodalanishi 1-jadvalda keltirilgan.

1-jadval

Oraliqlar turi	Geometrik tasviri	Belgilanishi	Tengsizliklar yordamida ifodalanishi
Interval		(a,b)	$a<x<b$
Kesma (segment)		$[a,b]$	$a\le x\le b$
Yarim interval		$(a,b]$	$a<x\le b$
Yarim interval		$[a,b)$	$a\le x<b$
Nur		$[a,+\infty)$	$x\ge a$
Nur		$(-\infty,b]$	$x\le b$
Ochiq nur		$(a,+\infty)$	$x>a$
Ochiq nur		$(-\infty,b)$	$x<b$

Eslatma. Amalda “interval”, “kesma (segment)”, “yarim interval”, “nur” terminlari yagona nom bilan “sonli oraliqlar” deb yuritiladi.

3.22. Haqiqiy sonning moduli

Haqiqiy son a ning ***moduli*** deb, agar $a \ge 0$ bo‘lsa, bu sonning o‘ziga, agar $a < 0$ bo‘lsa, unga qarama-qarshi son $-a$ ga aytiladi. a sonning moduli $|a|$ kabi belgilanadi. Demak,

$$|a| = \begin{cases} a, & agar \quad a \ge 0, \\ -a, & agar \quad a < 0. \end{cases}$$

Masalan, $|-5| = -(-5) = 5$, chunki $-5 < 0$; $|e-2| = e-2$ chunki, $e \approx 2{,}7... > 2$.

Geometrik nuqtai nazardan $|a|$ son koordinata to‘g‘ri chizig‘ida a nuqtadan 0 nuqtagacha bo‘lgan masofani bildiradi (8-rasm).

8-rasm

3.23. Haqiqiy son absolyut qiymatining xossalari

$x \in R$ sonning absolyut qiymati quyidagicha aniqlanadi:

$$|x| = \begin{cases} x, & agar\ x \ge 0\ bo'lsa, \\ -x, & agar\ x < 0\ bo'lsa \end{cases}$$

Haqiqiy sonning absolyut qiymatining xossalarini keltiramiz:

1⁰. $x \in R$ son uchun

$$|x| \ge 0,\ |x| = |-x|,\ x \le |x|,\ -x \le |x|$$

munosabatlar o‘rinli. Bu munosabatlar sonning absolyut qiymati ta’rifidan kelib chiqadi.

2⁰. Agar $x \in R$ sonlar

$$|x| < a \quad (a > 0) \qquad (1)$$

tengsizlikni qanoatlantirsa, bunday x sonlar

$$-a < x < a \qquad (2)$$

tengsizliklarni ham qanoatlantiradi va aksincha. Boshqacha aytganda (1) va (2) tengsizliklar ekvivalent tengsizliklardir:

$$|x| < a \Longleftrightarrow -a < x < a.$$

3⁰. Agar $x \in R$ sonlar $|x| \le a\,(a > 0)$ tengsizlikni qanoatlantirsa, bunday x lar $-a \le x \le a$ tengsizliklarni ham qanoatlantiradi va aksincha, ya’ni

$$|x| \le a \Longleftrightarrow -a \le x \le a .$$

4⁰. Ikki $x \in R$ va $y \in R$ haqiqiy sonlar yig‘indisining absolyut qiymati bu sonlar absolyut qiymatlarining yig‘indisidan katta emas, ya’ni

$$|x+y| \le |x| + |y| .$$

5⁰. $x \in R,\ y \in R$ sonlar uchun

$$|x-y| \ge |x| - |y|$$

tengsizlik o‘rinli.

6⁰. $x \in R,\ y \in R$ sonlar uchun

$$|x \cdot y| = |x| \cdot |y|$$

tenglik o‘rinli.

7⁰. $x \in R,\ y \in R,\ y \ne 0$ sonlar uchun

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|},\quad \big||x| - |y|\big| \le |x-y|$$

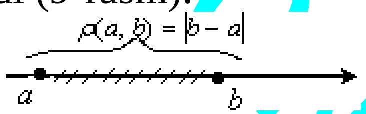
munosabatlar o‘rinli.

3.24. Koordinata to‘g‘ri chizig‘idagi ikki nuqta orasidagi masofa

Agar a va b-koordinata to‘g‘ri chizig‘idagi ikkita nuqta bo‘lsa, u holda ular orasidagi $\rho(a,b)$ masofa

$$\rho(a,b) = |b-a|$$

formula orqali ifodalanadi (9-rasm).



9-rasm

Masalan, $a=-3, b=-7$ bo‘lsa, bu nuqtalar orasidagi masofa

$$\rho(-3,-7) = |-7-(-3)| = |-7+3| = 4 .$$

3.25. To‘plamning aniq yuqori va aniq quyi chegaralari. Chegaralangan to‘plam

Ta’rif. Agar shunday M son mavjud bo‘lsaki, ixtiyoriy $x \in E$ uchun $x \le M$ tengsizlik bajarilsa, E $(E \subset R)$ to‘plam yuqoridan chegaralangan deyiladi, M son esa E ning ***yuqori chegarasi*** deyiladi.

Bu ta’rifni qisqacha quyidagicha aytsa ham bo‘ladi, agar shunday $M \in R$ va ixtiyoriy $x \in E$ uchun $x \le M$ bo‘lsa, E $(E \subset R)$ to‘plam ***yuqoridan*** chegaralangan deyiladi, M son esa E ning yuqori chegarasi deyiladi.

Ta’rif. Agar

$$\forall M \in R, \exists x_0 \in E : x_0 > M$$

bo‘lsa, E $(E \subset R)$ to‘plam ***yuqoridan chegaralanmagan*** bo‘ladi

Ta’rif. Agar

$$\exists m \in R, \forall x \in E : x \geq m$$

bo‘lsa, E $(E \subset R)$ to‘plam quyidan chegaralangan deyiladi, m son esa E $(E \subset R)$ ning ***quyi chegarasi*** deyiladi.

Ta’rif. Agar

$$\forall m \in R, \exists x_0 \in E : x_0 < m$$

bo‘lsa, E to‘plam quyidan chegaralanmagan deyiladi.

Ta’rif. Agar $E \subset R$ to‘plam ham quyidan, ham yuqoridan chegaralangan bo‘lsa, E to‘plam ***chegaralangan*** deyiladi.

Masalan, $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ to‘plam chegaralangan, $N = \{1,2,3,\ldots\}$ to‘plam quyidan chegaralangan, yuqoridan chegaralanmagan; $F = \{x : x \in R, x < 0\}$ to‘plam esa quyidan chegaralanmagan, yuqoridan chegaralangan to‘plam bo‘ladi.

Misol. Ushbu $A = \left\{n : n \in N, \frac{n}{n^2+1}\right\}$ to‘plamning chegaralanganligi ko‘rsatilsin.

Ravshanki, ixtiyoriy $n \in N$ da $\frac{n}{n^2+1} > 0$ bo‘ladi. Demak, A to‘plam quyidan chegaralangan.

Agar

$$0 \leq (n-1)^2 = n^2 - 2n + 1 \Rightarrow 2n \leq n^2 + 1 \Rightarrow \frac{n}{n^2+1} \leq \frac{1}{2}$$

bo‘lishini e’tiborga olsak, unda A to‘plamning yuqoridan chegaralanganligini topamiz.

Agar E to‘plam yuqoridan chegaralangan bo‘lsa, uning yuqori chegarasi cheksiz ko‘p bo‘ladi. Bu tasdiq M sondan katta bo‘ladi ya’ni har qanday haqiqiy son E to‘plamning yuqori chegarasi bo‘la olishidan kelib chiqadi.

Shuningdek, agar E to‘plam quyidan chegaralangan bo‘lsa, uning quyi chegarasi ham cheksiz ko‘p bo‘ladi. Bu esa m sondan kichik bo‘lgan har qanday haqiqiy son E to‘plamning quyi chegarasi bo‘la olishidan kelib chiqadi.

Teorema. Har qanday yuqoridan chegaralangan to‘plam uchun uning yuqori chegaralar orasida eng kichigi mavjud.

E $(E \subset R)$ to‘plam yuqoridan chegaralangan bo‘lsin, ya’ni shunday haqiqiy M son mavjudki, ixtiyoriy $x \in E$ uchun $x \leq M$ tengsiz o‘rinli.

Ta’rif. Yuqoridan chegaralangan E to‘plamning yuqori chegaralarining eng kichigi E ning ***aniq yuqori chegarasi*** deb ataladi va $\sup E$ kabi belgilanadi.

Misol. $E = \{x : x < 0\}$: $\sup E = 0$.

Teorema. Har qanday quyidan chegaralangan to‘plam uchun quyi chegaralari orasida eng kattasi mavjud.

Ta’rif. Quyidan chegaralangan E to‘plamning quyi chegaralarining eng kattasi E ning aniq quyi chegarasi deb ataladi va $\inf E$ kabi belgilanadi.

Misol. $E = \left\{\frac{1+n}{n}, n = 1, 2, \ldots, n, \ldots\right\}$, $\inf E = 1$.

M1. To’plamlar, haqiqiy sonlar va induksiyaga doir mashqlar.

1. Agar A={1,2,3,4,7,8,9,11,17,19}, B={2,4,8,12,13,17,19,21}, C={0,1,3,7,8,11,13,22,21} bo’lsa, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$, $A \backslash B$, $B \backslash C$, $A \cap B$, $A \cap C$, $B \cap C$, $A \cap B \cap C$ topilsin.

2. Agar A={5,7,9,11,13} va C={1,2,3,4,5,6,9,10,11,12,13,14} bo’lsa, A to’plamning B to’plamga to’ldiruvchi C_sA to’plam topilsin.

3. Agar A={1,2,5,6,7,8,9}, B={1,2,3,4,5,6,7,8,9,10} bo’lsa, simmetrik $A\Delta B$ ayirma to’plam topilsin.

4. Agar A={1,3,4}, B={2,3,4,5} bo’lsa, $A \times B$ Dekard ko’paytma topilsin.

5. Agar A=Z, B=Z bo’lsa, $A \times B$ tekislikda qanday elementlardan iborat bo’ladi?

6. Agar A={1,2,3}, B={1,2,3,4,5,6,7,8,9}, C={0,1,2,4,5} bo’lsa, A+B, A+C, B+C, A+B+C to’plamlar quvvati topilsin.

7. Agar D={x: |x-3|<4} to’plam element xususiyatlari bilan berilgan bo’lsa, to’plamga qancha butun son bor?

8. Agar A={a,b,c}, B={1,2,3}, C={k,l,m}, D={4,5,6} bo’lsa, qaysi to’plamlar orasida o’zaro bir qiymatli akslantirish mumkin?

9. Agar A={1,2,3,4,5,6,7,8,9,10} bo’lsa, qism to’plamlarini soni qancha va ularni toping.

10. Agar A={1,2,3,4,5} bo’lsa, uni bo’laklarga ajrating.

11. To’g’ri munosabatlarni toping:

1) $N \cup Z = Z$ 2) $N \cap Z = \emptyset$ 3) $N \cap Z = Z$

A) 1 B) 2 C)3 D) 1,2 E)T.J.Y

12. A={0,1,2,...,9}, N={1,2,3,...,n} bo'lsa, noto'g'ri munosabatni toping.

1) $A \cup N = N$ 2)$A \cup N = A$ 3)$A \backslash N = \{0\}$

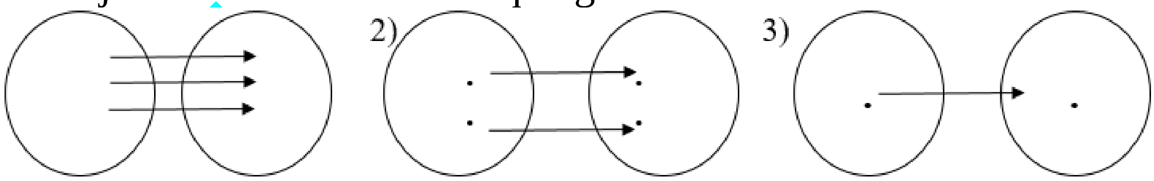
A) 1 B) 2 C) 3 D) 1,2 E)T.J.Y

13. A=[0,1] va B=[1,2] bo'lsa, A va B orasida qaysi funksiya o'zaro bir qiymatli akslantirish o'rnatadi.

1) y=x+1 2) y=2-x 3) y=1-x

A) 1 B) 2 C) 3 D) 1,2 E) T.J.Y

14. Bijektiv akslantirishni aniqlang.



15. y=|x+1| funksiyaning R^+ dagi siqilganini aniqlang.

1) y=|x+1| 2) y=-x-1 3) y=-x+1

A) 1 B) 2 C) 3 D) 1,2 E) T.J.Y

16. A={ax^2+bx+c, $x \in R$}, B={D=b^2-4ac≤ 0}, C={D$\leq 0, a < 0$} bo'lsa, to'g'ri teoremani aniqlang.

1)$A => B$ 2) $A <=> C$ 3) $B => C$

A) 1 B) 2 C) 3 D) 1,2 E) T.J.Y

17. Ratsional sonlar to'plamida qaysi tengsizlik o'rinli.

1) Agar a>b>0, c>d>0 bo'lsa, ac>bd bo'ladi;

2) Agar a>b, c<d bo'lsa, a-c>b-d bo'ladi;

3) Agar a>b>0 bo'lsa, $a^n>b^n$ (n$\in N$) bo'ladi.

A) 1 B) 2 C) 2,3 D)1,2,3 E) T.J.Y

18. Sonlarning qaysi biri chekli o'nli ratsional kasrni ifodalaydi?

1)$\frac{7}{55}$ 2)$\frac{17}{20}$ 3) $\frac{15}{23}$ 4) $\frac{47}{64}$ 5) $\frac{119}{125}$

A) 2,4 B) 2,4,5 C) 1,2 D) 1,3 E) T.J.Y

19. Qaysi javobga ratsional sonlar xossasi keltirilgan.

1) Chegaralanganlik, tartiblanganlik
2) Zichlik, chegaralanganlik
3) Tartiblanganlik, zichlik
4) Chegaralanganlik, zichlik(uzluksizlik),tartiblanganlik

A) 1,2 B) 2,3 C)4 D)1,3 E) T.J.Y

20. Qaysi to'plam ratsional sonlar to'plami bo'ladi?

1) Q={$\frac{m}{n}; m \in z, n \in z$} 2) Q= {$\frac{m}{n}; m \in N, n \in N$}

3) Q=$\{\frac{m}{n}; m \in N, n \in z\}$

A) 1 B) 2 C)3 D) 1,2 E)T.J.Y

21. $7{,}92 \times 0,(15) = ?$

A) 1 B) 2 C) 3 D) 1,2 E)T.J.Y

22. $\frac{1}{3}$ va $\frac{1}{2}$ ratsional sonlar orasida nechta ratsional son bor?

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

23. Ratsional sonlar to'plamida quyidagilarning qaysilari kesim tashkil qiladi?

1) A=$\{a \in Q: a \le -2\frac{1}{2}\}$, A\`=$\{a \in Q: a > -2(3)\}$,

2) A=$\{a \in Q: a < -2\frac{1}{2}\}$, A\`=$\{a \in Q: a \ge -2\frac{1}{3}\}$,

3) A=$\{a \in Q: a < \frac{7}{3}\}$, A\`=$\{a \in Q: a > \frac{7}{3}\}$.

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

24. Qaysi kesim irratsional kesimni ifodalaydi? ($r_0 \in Q$)

1) A=$\{ a \in Q: a < r_0\}$, A\`= $\{a \in Q: a \ge r_0\}$

2) A=$\{a \in Q: a \le r_0\}$, A\`=$\{a \in Q: a > r_0\}$

3) A=$\{a \in Q: a < r_0\}$, A\`=$\{a \in Q: a > r_0\}$

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

25. Noto'g'ri tasdiqni aniqlang.

1. a>b, c>d bo'lsa, ac>bd bo'ladi.
2. a>b, c>d bo'lsa, a-c>b-d bo'ladi.
3. a>b>0 bo'lsa, $a^n>b^n$ bo'ladi, $n \in N$

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

26. a – manfiy irratsional son, $[a+1]=a_0$ bo'lsin. Qaysi tenglik a sonining kami bilan o'nli yaqinlashish bo'ladi:

1. $a_0, a_1, a_2, \ldots a_n+10^{-n}$ 2. $a_0,a_1,a_2,a_3,\ldots,a_n-10^{-n}$
3. $a_0,a_1,a_2,a_3,\ldots,a_n\pm10^{-n}$

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

27. X={1+(-1)^n; n$\in N$} to'plam haqidagi to'g'ri tasdiqni toping:

1) Yuqoridan chegaralangan;
2) Quyidan chegaralangan;
3) Chegaralanmagan

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

28. X=(a,b) oraliq haqidagi to'g'ri tasdiqni toping.

1) supX=b; 2) infX=b; 3) ∃supX, ∃infX

A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

29. Quyidagi tengsizliklarning qaysi o'rinli?
   1) $|a|+|b| \geq |a+b|$ 2) $|a-b| \geq ||a|-|b||$ 3) $|a|-|b| \leq |a-b|$
   A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

30. To'g'ri tenglikni aniqlang:
   1) $\forall c \in R, \ c \times 0 = 0$ 2) $\forall c \in R; c^0=1$ 3) $\sqrt[n]{a^n}=a$
   A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

31. To'g'ri tenglikni aniqlang.
   1) $\sqrt[2n]{a^{2n}}=|a|$ 2) $\sqrt[2n+1]{a^{2n+1}}=a$ 3) $\sqrt[n]{a+b} = \sqrt[n]{a} + \sqrt[n]{b}$
   A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

32. Eng katta sonni aniqlang.
   1) $2^{\sqrt{2}}$ 2) $2^{1,4}$ 3) $2^{1,5}$
   A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

33. Eng kichik sonni aniqlang
   1) $2^{-\sqrt{2}}$ 2) $2^{-1,4}$ 3) $2^{-1,5}$
   A) 1 B) 2 C) 3 D)1,2 E) T.J.Y

34. Matematik induksiya usuli yordamida quyidagi tengliklarni isbot qiling.

1) $1+3+5+...+(2n-1) =n^2$

2) $1\cdot 2 + 2\cdot 5 + \cdots + n(3n-1) = n^2(n+1)$

3) $1+2+2^2+...+ 2^{n-1}=2^n-1$

4) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

5) $(1-\frac{1}{4})(1-\frac{1}{9})...(1-\frac{1}{(n+1)^2}) = \frac{n+2}{2n+2}$

6) Bernulli tengsizligini isbotlang.
   $(1+x_1)(1+x_2)...(1+x_n) \geq 1 + x_1+x_2+...+x_n$ bu yerda $x_1, x_2,...x_n$ – bir xil ishorali va -1 dan katta sonlar.

7) Agar $x>-1$ bo'lsa, $(1+x)^n \geq 1 + nx$ $(n>1)$, tengsizlikni isbot qiling

8) $1^2+2^2+3^2+...+n^2= \frac{n(n+1)(2n+1)}{6}$

9) Tengsizlikni isbotlang
   $n!<(\frac{n+1}{2})^n$, $n>1$

10) $1+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, \ (n \geq 2)$

11) n^5-n ifodani 5 ga karraliligini isbotlang.

4§. SONLAR KETMA-KETLIGI VA UNING LIMITI

4.1. Sonli ketma-ketlik tushunchasi

Faraz qilaylik, f har bir natural son $n \in N$ ga biror haqiqiy $x_n \in R$ sonni mos qo‘yuvchi akslantirish bo‘lsin:

$$f : n \to x_n \quad \left(x_n = f(n)\right).$$

Bu akslantirish qiymatlaridan tuzilgan

$$x_1, x_2, x_3, ..., x_n, ...$$

ifoda ***haqiqiy sonlar ketma-ketligi*** (qisqacha ***sonlar ketma-ketligi***) deyiladi va $\{x_n\}$ ko‘rinishida belgilanadi.

Misol. $x_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$

4.2. Sonli ketma-ketlikning yuqoridan (quyidan) chegaralanganligi

Ta’rif. Agar shunday o‘zgarmas M soni mavjud bo‘lsaki, $\forall n \in N$ uchun $x_n \le M$ $\left(x_n \ge M\right)$ bo‘lsa, u holda $\{x_n\}$ ketma-ketlik ***yuqoridan (quyidan) chegaralangan*** deyiladi.

Bu ta’rifni quyidagicha qisqacha kabi ifodalash mumkin:

$$\exists M \in R, \quad \forall n \in N : x_n \le M \ \left(x_n \ge M\right).$$

Misol. 1) 0, $-1, -2, -3, ..., -n, ...$ ketma-ketlik yuqoridan chegaralangan, chunki ketma-ketlikning har bir hadi 0 dan katta emas. 2) $1!, 2!, 3!, ..., n!, ...$ ketma-ketlik quyidan chegaralangan, chunki $\{n!\}$ ketma-ketlikning har bir hadi 1 dan kichik emas.

4.3. O‘suvchi sonli ketma-ketliklar

Ta’rif. Agar ixtiyoriy $n \in N$ uchun

$$x_n \le x_{n+1} \quad \left(x_n < x_{n+1}\right)$$

tengsizlik o‘rinli bo‘lsa, $\{x_n\}$ ***o‘suvchi (qat’iy o‘suvchi)*** ketma-ketlik deyiladi.

Misol. 1, 2, 2, 3, 3, 3, …. o‘suvchi; $2, 2^2, 2^3, ..., 2^n, ...$ qat’iy o‘suvchi ketma-ketliklar bo‘ladi.

4.4. Kamayuvchi sonli ketma-ketliklar

Ta’rif. Agar ixtiyoriy $n \in N$ uchun

$$x_n \ge x_{n+1} \quad \left(x_n > x_{n+1}\right)$$

tengsizlik o‘rinli bo‘lsa, $\{x_n\}$ ***kamayuvchi*** **(*qat'iy kamayuvchi*)** ketma-ketlik deyiladi.

Misol. $1, \frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2n}, ...$ kamayuvchi.

4.5. Monoton ketma-ketliklar

Ta'rif. O‘suvchi va kamayuvchi ketma-ketliklar umumiy nom bilan ***monoton ketma-ketlilar*** deyiladi.

4.6. Sonli ketma-ketlikning limiti

$$x_1, x_2, x_3, ..., x_n, ...$$

sonlar ketma-ketligi hamda biror a son ($a \in R$) berilgan bo‘lsin.

Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday natural $n_0 = n_0(\varepsilon)$ son topilsaki, barcha $n > n_0$ natural sonlar uchun

$$|x_n - a| < \varepsilon$$

tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik ***yaqinlashuvchi*** deyiladi, a son esa $\{x_n\}$ ***ketma-ketlikning limiti*** deyiladi va

$$\lim_{n\to\infty} x_n = a \text{ yoki } x_n \underset{n\to\infty}{\to} a$$

kabi belgilanadi.

Bu ta’rifni qisqacha

$$\forall\, \varepsilon > 0, \exists\, n_0 = n_0(\varepsilon) \in N, \quad \forall n > n_0 \Rightarrow |x_n - a| < \varepsilon$$

kabi ifodalash mumkin.

Misol. Ushbu $x_n = \sqrt[n]{a} \quad (a > 1, \sqrt[n]{a} > 1) : a, \sqrt{a}, \sqrt[3]{a}, ..., \sqrt[n]{a}, ...$ketma-ketlikning limiti 1ga teng bo‘lishini ko‘rsating.

Ixtiyoriy musbat ε sonni olamiz. Olingan ε songa ko‘ra natural n_0 sonni $n_0 = \left[\frac{\lg a}{\lg(1+\varepsilon)}\right] + 1$ bo‘lsin deb qaraylik. Bu holda $n > n_0$ tengsizlikni qanoatlantiruvchi barcha natural n sonlar uchun

$$|x_n - 1| = |\sqrt[n]{a} - 1| = \sqrt[n]{a} - 1 < a^{\frac{1}{n_0}} - 1 < a^{\frac{\lg(1+\varepsilon)}{\lg a}} - 1 = \left(a^{\log_a 10}\right)^{\lg(1+\varepsilon)} - 1 = \varepsilon$$

munosabat o‘rinli bo‘ladi. Bu esa berilgan ketma-ketlikning limiti 1 ga teng bo‘lishini ko‘rsatadi.

4.7. Yaqinlashuvchi ketma-ketliklarning xossalari

1°. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va $\lim_{n\to\infty} x_n = a$ bo‘lib, $a > p$ $(a < q)$ bo‘lsa, u holda ketma-ketlikning biror hadidan boshlab keyingi barcha hadlari ham p sondan katta (q sondan kichik) bo‘ladi.

2°. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, u chegaralangan bo‘ladi.

3°. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, uning limiti yagonadir.

4.8. Cheksiz kichik miqdor

Ta’rif. Agar

$$\lim_{n\to\infty} x_n = 0$$

bo‘lsa, u holda x_n ga cheksiz kichik miqdor deyiladi.

Misol. $x_n = \frac{1}{n}$ o‘zgaruvchi cheksiz kichik miqdordir, chunki $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Cheksiz kichik miqdorlar α_n, β_n kabi belgilanadi.

1-lemma. Chekli sondagi cheksiz kichik miqdorlar yig‘indisi cheksiz kichik bo‘ladi.

2-lemma. Chegaralangan ketma-ketlik bilan cheksiz kichik miqdorning ko‘paytmasi cheksiz kichik bo‘ladi.

4.9. Cheksiz katta miqdor

Ta’rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday natural n_0 son topilsaki, ixtiyoriy $n > n_0$ uchun

$$|x_n| > \varepsilon$$

bo‘lsa, x_n ga cheksiz katta miqdor deyiladi.

Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday $n_0 \in N$ son topilsaki, $\forall\, n > n_0$ uchun $x_n > \varepsilon$ ($x_n < -\varepsilon$) bo‘lsa, unda $\{x_n\}$ ketma-ketlikning limiti $+\infty$ ($-\infty$) deb olinadi va $\lim_{n\to\infty} x_n = +\infty$, ($\lim_{n\to\infty} x_n = -\infty$) kabi belgilanadi.

Misol. Ushbu $\{(-1)^n \cdot n\}: -1, 2, -3, 4, \ldots, (-1)^n \cdot n, \ldots$ ketma-ketlik cheksiz katta bo‘ladi. Haqiqatan, $|(-1)^n \cdot n| = n$ bo‘lib, har qanday musbat M son olinganda ham $n \in N$ sonni shunday tanlab olish mumkinki, $|(-1)^n \cdot n| = n > M$ bo‘ladi.

4.10. Limitga ega bo‘lgan ketma-ketliklar haqida teoremalar (yaqinlashuvchi ketma-ketliklar ustida arifmetik amallar)

Aytaylik, $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar berilgan bo‘lib, ular chekli limitga ega bo‘lsin:

$$\lim_{n\to\infty} x_n = a, \ \lim_{n\to\infty} x_n = b.$$

U holda:

1) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n = a \pm b$;

2) $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n = a \cdot b$;

3) $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n} = \dfrac{a}{b}$ $(b \neq 0,\ y_n \neq 0)$

bo'ladi.

4) agar ixtiyoriy $n > N$ uchun $x_n \le y_n$ bo'lsa, $a \le b$ bo'ladi.

5) e **soni.** Quyidagi $\left(1+\dfrac{1}{n}\right)^n$ $(n = 1,2,3,\ldots)$ ketma-ketlik $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e = 2{,}718281\ldots$ chekli limitga ega. e irratsional son.

4.11. Ketma-ketlik limitining mavjudligi

1. Ixtiyoriy $n > n_0$ uchun $y_n \le x_n \le z_n$ tengsizlik o'rinli bo'lib, $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = a$ bo'lsa, $\lim\limits_{n\to\infty} x_n = a$.

2. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, $\forall n \in N$ uchun $x_n \le c$ $(x_n \ge c)$ bo'lsa, u holda $\lim x_n \le c$ $(\lim x_n \ge c)$ bo'ladi.

3. Agar $\{x_n\}$ ketma-ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, u chekli limitga ega. Agar yuqoridan chegaralangan bo'lmasa uning limiti $+\infty$ bo'ladi.

4. Agar $\{x_n\}$ ketma-ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, u chekli limitga ega: $\{x_n\}$ ketma-ketlik quyidan chegaralangan bo'lmasa uning limiti $-\infty$ bo'ladi.

5. <u>Ichma-ich joylashgan segmentlar prinsipi:</u> ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketlik berilgan bo'lib quyidagilar:

1) $\{x_n\}$ o'suvchi, $\{y_n\}$ kamayuvchi ketma-ketlik;

2) ixtiyoriy $n \in N$ lar uchun $x_n < y_n$;

3) $\lim\limits_{n\to\infty}(y_n - x_n) = 0$ bajarilgan bo'lsa, $\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi va $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$ o'rinli bo'ladi.

6. Monoton va chegaralangan ketma-ketlik limitga ega.

4.12. Qismiy ketma-ketlik

Faraz qilaylik, ixtiyoriy $x_1, x_2, x_3, ..., x_n, ...$ ketma-ketlik berilgan bo‘lsin. Bu ketma-ketlik hadlaridan tuzilgan ushbu $x_{n_1}, x_{n_2}, x_{n_3}, ..., x_{n_k}, ...$ ($n_1 < ... < n_k < ...,\ \ n_k \in N,\ k = 1,2,...,$) ketma-ketlik berilgan $\{x_n\}$ ketma-ketlikning ***qismiy ketma-ketligi*** deyiladi

Teorema. Agar $\{x_n\}$ ketma-ketlik limitga (chekli yoki cheksiz) ega bo‘lsa, uning har qanday qismiy ketma-ketligi shu limitga ega bo‘ladi.

Eslatma. Ketma-ketlik qismiy ketma-ketligining limiti mavjud bo‘lishidan berilgan ketma-ketlik limiti mavjud bo‘lishi har doim ham kelib chiqavermaydi.

Ta’rif. $\{x_n\}$ ketma-ketlikning yaqinlashuvchi qismiy ketma-ketligining limiti berilgan ***ketma-ketlikning qismiy limiti*** deyiladi.

Misol. $1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}, ...$ ketma-ketlik, $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$ ketma-ketlikning qismiy ketma-ketligidir.

Lemma (*Bolsano-Veyershtrass*). Agar $\{x_n\}$ chegaralangan bo‘lsa, bu ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin.

4.13. Fundamental ketma-ketliklar. Koshi mezoni

Ta’rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday $n_0 \in N$ son mavjud bo‘lsaki, barcha $n > n_0$ va $m > n_0$ lar uchun

$$|x_n - x_m| < \varepsilon$$

tengsizlik bajarilsa, $\{x_n\}$ ***fundamental ketma-ketlik*** deyiladi.

Teorema (*Koshi mezoni*). Ketma-ketlik yaqinlashuvchi bo‘lishi uchun u fundamental bo‘lishi zarur va yetarlidir.

4.14. Ketma-ketlikning quyi va yuqori limitlari

Ta’rif. $\{x_n\}$ ketma-ketlik qismiy limitlarining eng kattasi berilgan ***ketma-ketlikning yuqori limiti*** deyiladi va u

$$\overline{\lim_{n\to\infty}} x_n$$

ko‘rinishida belgilanadi.

Ta’rif. $\{x_n\}$ ketma-ketlik qismiy limitlarining eng kichigi berilgan ***ketma-ketlikning quyi limiti*** deyiladi va u

$$\underline{\lim_{n\to\infty}} x_n$$

ko'rinishida belgilanadi.

Teorema. Har qanday ketma-ketlikning quyi va yuqori limiti mavjud.

Teorema. $\{x_n\}$ ketma-ketlik c limitga ega bo'lishi uchun $\underline{\lim_{n\to\infty}} x_n = \overline{\lim_{n\to\infty}} x_n = c$ bo'lishi zarur va yetarli.

Misol. Ushbu $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$ ketma-ketlikning yuqori limiti $\overline{\lim_{n\to\infty}} x_n = 3$, quyi limiti esa $\underline{\lim_{n\to\infty}} x_n = 1$ bo'ladi.

4.15. Ayrim ketma-ketlik (funksiya) lar limitlari

1. $\lim\limits_{n\to\infty} q^n = 0 \quad (|q|<1)$.

◀ $\dfrac{1}{|q|^n} = \left(1+\dfrac{1-|q|}{|q|}\right)^n = 1+n\dfrac{1-|q|}{|q|} + \dfrac{n(n-1)}{2!}\left(\dfrac{1-|q|}{|q|}\right)^2 + \ldots + \left(\dfrac{1-|q|}{|q|}\right)^n > n\cdot\dfrac{1-|q|}{|q|} \Rightarrow$

$\dfrac{1}{|q|^n} > n\cdot\dfrac{1-|q|}{|q|}$, $\quad |q|^n = |q^n| < \dfrac{|q|}{1-|q|}\cdot\dfrac{1}{n} < \varepsilon$ bu $\forall\varepsilon>0 \quad n > \dfrac{|q|(1-|q|)^{-1}}{\varepsilon}$ da o'rinli.▶

2. $\lim\limits_{n\to\infty}\dfrac{n}{2^n} = 0 \quad \left[\lim\limits_{n\to\infty}\dfrac{2^n}{n} = +\infty\right]$.

◀ $\quad \left|\dfrac{n}{2^n}\right| = \dfrac{n}{2^n} = \dfrac{n}{(1+1)^n} = \dfrac{n}{1+n+\dfrac{n(n-1)}{2}+\ldots+1} < \dfrac{n}{\dfrac{n(n-1)}{2}} = \dfrac{2}{n-1} < \varepsilon, \ \forall\varepsilon>0 \quad n > 1+\dfrac{2}{\varepsilon}$

bo'lganda. ▶

2.1*. $\lim\limits_{n\to\infty}\dfrac{n}{a^n} = 0 \ (a>1) \quad \left[\lim\limits_{n\to\infty}\dfrac{a^n}{n} = +\infty\right]$.

◀ $a>1, a=1+\lambda, \lambda>0$.

$\left|\dfrac{n}{a^n}\right| = \dfrac{n}{a^n} = \dfrac{n}{(1+\lambda)^n} = \dfrac{n}{1+n\lambda+\dfrac{n(n-1)}{2}\lambda^2+\ldots+\lambda^n} < \dfrac{n}{\dfrac{n(n-1)}{2}\lambda^2} =$

$= \dfrac{2}{(n-1)\lambda^2} < \varepsilon$. $n>2$ da $n-1 > \dfrac{n}{2} \quad \dfrac{2}{(n-1)\lambda^2} < \dfrac{4}{n\lambda^2} < \varepsilon$.▶

3. $\lim\limits_{n\to\infty}\dfrac{n^k}{a^n} = 0 \ (a>1, k>0) \quad \left[\lim\limits_{n\to\infty}\dfrac{a^n}{n^k} = +\infty\right]$.

◀ $a>1, \lambda>0, a=1+\lambda$. Binomdan

$a^n=(1+\lambda)^n=1+n\lambda+\frac{n(n-1)}{2}\lambda^2+\ldots+\lambda^n>\frac{n(n-1)}{2}\lambda^2$. $n>2$ da $n-1>\frac{n}{2}$ natijada $a^n>\frac{n^2}{4}\lambda^2$ yoki $\lambda=a-1\Rightarrow a^n>\frac{(a-1)^2}{4}n^2$. $\frac{a^n}{n}>\frac{(a-1)}{4}n$, $n\to\infty$ $\left(\frac{a-1}{4}\right)n\to\infty$ $\frac{a^n}{n}\to\infty\Rightarrow\frac{n}{a^n}\to 0$ bu $k=1$ hol (3) kabi, yoki $\lim\limits_{n\to\infty}\frac{a^n}{n}=+\infty$ isbot bo‘ldi.

Endi $k>1$ bo‘lsin. $\frac{a^n}{n^k}=\left[\frac{\left(a^{\frac{1}{k}}\right)^n}{n}\right]^k>\frac{\left(a^{\frac{1}{k}}\right)^n}{n}$ $\left[b=a^{\frac{1}{k}}>1 \quad \frac{b^n}{b}\to\infty\right]\Rightarrow\lim\limits_{n\to\infty}\frac{a^n}{n^k}=+\infty$.▶

4. $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$. ◀ $a=1$ $\sqrt[n]{1}=1$. 1) $a>1$ bo‘lsin. Unda $\sqrt[n]{a}>1$ bo‘lib

$a=\left[1+(\sqrt[n]{a}-1)\right]^n=$

$=1+n(\sqrt[n]{a}-1)+\frac{n(n-1)}{2!}(\sqrt[n]{a}-1)^2+\ldots+(\sqrt[n]{a}-1)^n>n(\sqrt[n]{a}-1)\Rightarrow a>n(\sqrt[n]{a}-1)\Rightarrow 0<\sqrt[n]{a}-1<$

$<\frac{a}{n}<\varepsilon$ $n>\frac{a}{\varepsilon}$ $(\varepsilon>0)$ da $\sqrt[n]{a}\to 1$, $n\to\infty$.▶

2) $0<a<1$, unda $\frac{1}{a}>1$. 1) dan $\sqrt[n]{\frac{1}{a}}\to 1$ ya’ni, $n\to\infty$.

Bu holda $\lim\limits_{n\to+\infty}\sqrt[n]{a}=\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{\frac{1}{a}}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{a}}}=1$.

5. $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$.

◀ $n=\left[1+(\sqrt[n]{n}-1)\right]^n=1+n(\sqrt[n]{n}-1)+\frac{n(n-1)}{2!}(\sqrt[n]{n}-1)^2+\ldots+(\sqrt[n]{n}-1)^n$ bundan $n>\frac{n(n-1)}{2!}(\sqrt[n]{n}-1)^2$ bundan $(\sqrt[n]{n}-1)^2<\frac{2}{n-1}\Rightarrow\left|\sqrt[n]{n}-1\right|<\sqrt{\frac{2}{n-1}}<\varepsilon$ bo‘ladi. $n>1+2\varepsilon^{-2}$ o‘rinli bo‘lsa, $\left|\sqrt[n]{n}-1\right|\to 0$.▶

6. $\lim\limits_{n\to\infty}\frac{a^n}{n!}=0$.

◀ $\frac{a^n}{n!}=\frac{\overbrace{a\cdot a\cdot a\cdot\ldots\cdot a}^{n\ ta}}{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\cdot\ldots\cdot n}$. $0<\left|\frac{a^n}{n!}\right|=\frac{|a|}{1}\cdot\frac{|a|}{2}\cdot\ldots\cdot\frac{|a|}{m}\cdot\frac{|a|}{m+1}\cdot\ldots\cdot\frac{|a|}{n}<\frac{|a|^m}{m!}\left(\frac{|a|}{m+1}\right)^{n-m}<$

$<\varepsilon$: $n\to\infty$, $\frac{|a|^{n-m}}{m+1}\to 0$ yetarlicha katta n larda.▶

6*. $\lim\limits_{n\to\infty}\frac{2^n}{n!}=0$. ◀ $0<\frac{2^n}{n!}=\frac{\overbrace{2\cdot 2\cdot 2\cdot\ldots\cdot 2}^{n\ ta}}{1\cdot 2\cdot 3\cdot\ldots\cdot n}\le 2\cdot\left(\frac{2}{3}\right)^{n-2}=\frac{9}{2}\cdot\left(\frac{2}{3}\right)^n$, $n\to\infty$, $\left(\frac{2}{3}\right)^n\to 0$.▶

7. $\lim\limits_{n\to\infty}nq^n=0$.

◀ Agar $|q|<1$ bo‘lsa. $\left|\frac{1}{q}\right|=a>1$ $|n\cdot q^n|=\frac{n}{\left|\frac{1}{q}\right|^n}=\frac{n}{a^n}$ 2* va 3 dagidek isbot bo‘ladi.▶

8. $\lim\limits_{n\to\infty}\frac{\log_b n}{n}=0 \quad (b>1)$.

◀ $\lim\limits_{n\to\infty}\frac{n}{a^n}=0$ 2* dan. Unda $\frac{1}{a^n}<\frac{n}{a^n}<1$ (yetarli katta n larda) $a>1$ va $\forall\varepsilon>0$ uchun $a=b^\varepsilon$ bo‘lsin $(a=b^\varepsilon>1)$. Natijada $\frac{1}{a^n}<\frac{n}{a^n}<1$ dan $\frac{1}{b^{\varepsilon n}}<\frac{n}{b^{\varepsilon n}}<1$ ikkala tomonini $b^{\varepsilon n}$ ga ko‘paytiramiz. $1<n<b^{\varepsilon n}$ va buni "b" asosga ko‘ra logarfmlaymiz. $\log_b 1<\log_b n<\log_b b^{\varepsilon n}\Rightarrow 0<\log_b n<\varepsilon n$ yoki n ga bo‘lsak $0<\frac{\log_b n}{n}<\varepsilon$ isbot bo‘ldi .▶

9. $\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n}}=0$. ◀ $n!>\left(\frac{n}{3}\right)^n$ ekanligini ko‘rsatamiz.▶.

Matematik induksiyada: $n=1$ da $1>\frac{1}{3}$ o‘rinli. n da o‘rinli $n!>\left(\frac{n}{3}\right)^n$. $n\sim n+1$ da o‘rinliligini ko‘rsatamiz.

$$(n+1)!=n!(n+1)>\left(\frac{n}{3}\right)^n(n+1)=\left(\frac{n+1}{3}\right)^{n+1}\left(\frac{3}{n+1}\right)^{n+1}\left(\frac{n}{3}\right)^n(n+1)=\left(\frac{n+1}{3}\right)^{n+1}\frac{3}{(n+1)^n}n^n=$$

$$=\left(\frac{n+1}{3}\right)^{n+1}\frac{3}{\left(\frac{n+1}{n}\right)^n}=\left(\frac{n+1}{3}\right)^{n+1}\frac{3}{\left(\frac{n+1}{n}\right)^n}>\left(\frac{n+1}{3}\right)^n.$$

Chunki,

$$\left(1+\frac{1}{n}\right)^n=1+\frac{n}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\ldots+\frac{n(n-1)(n-2)\ldots(n-n+1)}{n!}\frac{1}{n^n}=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+$$

$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)<1+1+\frac{1}{2!}+\ldots+\frac{1}{n!}<1+1+\frac{1}{2}+\ldots+\frac{1}{2^{n+1}}<1+1+\frac{1}{2}+\ldots+$$

$+\frac{1}{2^{n-1}}+\ldots=1+\frac{1}{1-\frac{1}{2}}=3$. Limit mavjudligi quyidagi tengsizlikdan kelib chiqadi. $0<\frac{1}{\sqrt[n]{n!}}<\frac{1}{\sqrt[n]{\left(\frac{n}{3}\right)^n}}=\frac{3}{n}<\varepsilon$ bu o‘rinli $n>\frac{3}{\varepsilon}$.▶

10. $x\to 0$ da $x\sin\sqrt{x}=x^{\frac{3}{2}}+o\left(x^{\frac{3}{2}}\right)$

M2. Sonlar ketma-ketligi va uning limitiga doir mashqlar

1. a, b sonlardan qaysi biri $\{x_n\}$ ketma-ketlikning hadlari bo'ladi:

1) $a = 1215$, $b = 12555$; $x_n = 5 \cdot 3^{2n-3}$

2) $a = 6$, $b = 11$; $x_n = \frac{(n^2+1)}{n+1}$.

3) $a = 248$, $b = 2050$; $x_n = 2^n - n$

2. Quyidagi ketma-ketliklarning umumiy hadini toping.

1) $\{8; 14; 20; 26; 32; \dots\}$.

2) $\{-0{,}5; 1{,}5; \ -4{,}5; 13{,}5; \ -40{,}5; \dots\}$

3) $\{2; 1; 0; \ -1; \ -2; \dots\}$

4) $\{7; 9; 13; 21; 37; \dots\}$

5) $\{2; 3; 7; 25; 121; \dots\}$

3. Quyidagi ketma-ketliklarning eng katta hadini toping.

1) $x_n = \frac{42}{3n^2-14n-17}$

2) $x_n = \frac{2n}{n^2+9}$

3) $x_n = 2^{-n} - 3 * 4^{-n}$

4) $x_n = \frac{\sqrt{n}}{150+n}$

4. Quyidagi ketma-ketliklarning eng kichik hadini toping

1) $x_n = (3n-5)(3n-11)$

2) $x_n = \log_3 2 - 3\log_3 n$

3) $x_n = n^2 - 9n - 100$

4) $x_n = n + \frac{5}{n}$

5. Quyidagi ketma-ketliklarning chegaranganligini ko'rsating.

1) $\left\{\frac{3n^2-1}{3+n^2}\right\}$ 2) $\left\{\frac{1-n}{\sqrt{n^2+1}}\right\}$

3) $\left\{\frac{n+(-1)^n}{3n-1}\right\}$ 4) $\left\{\frac{n^2+4n+8}{(n+1)^2}\right\}$

6. Quyidagi ketma-ketliklarning chegaranlanmaganligini ko'rsating.

1) $\{(-1)^n n\}$ 2) $\{n^2 - n\}$

3) $\left\{\frac{n^3}{n^2+1}\right\}$ 4) $\{n + (-1)^n n\}$

7. Quyidagi ketma-ketlikning qandaydir hadidan boshlab monoton ekanligini korsating.

1) $\left\{\frac{3n+4}{n+2}\right\}$ 2) $\{\sqrt{n+2} - \sqrt{n+1}\}$

3) $\{3^n - 2^n\}$ 4) $\{lg(n+1) - lgn\}$

8. Umumiy hadlari berilgan quyidagi ketma-ketliklarning limitini hisoblang.

1) $x_n = \frac{(n+1)^2}{5n^2}$ 2) $x_n = \frac{(-1+x)^3+(1-x)^3}{(n+1)^2+(n-1)^2}$

3) $x_n=\sqrt{n+1}-\sqrt{n}$ 4) $x_n=\frac{\sqrt[3]{n^2+2n-1}}{n+2}$

5) $x_n=\frac{1}{n^2}(1+2+3+\ldots.+n)$ 6) $x_n=\frac{1+2+3+\ldots.+n}{n+2}-\frac{n}{2}$

7) $x_n=\frac{1}{n^2}+\frac{2}{n^2}+\ldots.+\frac{n-1}{n^2}$ 8) $x_n=\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots.+\frac{(n-1)^2}{n^3}$

9) $x_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots.+\frac{1}{n(n+1)}$ 10) $x_n=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots.+\frac{1}{(2n-1)(2n+1)}$

9. Koshi mezonidan foydalanib quyidagi ketma-ketliklarni yaqinlashuvchilikka tekshiring.

1) $x_n=\frac{sin1}{2}+\frac{sin2}{2^2}+\ldots.+\frac{sinn}{2^n}$

2) $x_n=\frac{\cos 1!}{1\cdot 2}+\frac{\cos 2!}{2\cdot 3}+\ldots\frac{\cos n!}{n(n+1)}$

3) $x_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$

10. $\{x_n\}$ ketma-ketlik uchun $\inf\{x_n\}, sup\{x_n\}, \underline{lim}\{x_n\}, \overline{lim}x_n$ larni toping.

1) $x_n=1-\frac{1}{2}$ 2) $x_n=\frac{(-1)^n}{n}+\frac{1+(-1)^n}{2}$

3) $x_n=1+\frac{n}{n+1}\cos\frac{\pi n}{2}$ 4) $x_n=(-1)^n\frac{3n-1}{n+2}$

11. Quyidagi ketma-ketliklarni limitini toping

a) $u_n=n\sin\frac{\pi n}{2}$ b) $u_n=\frac{\sin\frac{\pi n}{2}}{lgn}$ (n> 1)

12. $x_n=\frac{n-1}{n+1}$ ketma-ketlikning qaysi nomeridan boshlab $|x_{n-1}|<10^{-4}$ tengsizlik o'rinli bo'ladi.

TESTLAR

13. $\lim_{n\to\infty}\sqrt[n]{n+1}=?$

A)1 B)2 C)3 D)1,2 E) 1,3

14. $\lim_{n\to\infty}\frac{n-1}{n}=a$ va $\lim_{n\to\infty}\frac{n+1}{n}=b$ bo'lsa to'g'ri munosabatlarni toping

1) $a<b$ 2) $a>b$ 3) $a=b$

A) 1 B) 2 C) 3 D) 1,2 E) 1,3

15. Tog'ri tenglikni toping

1) $\lim_{n\to\infty}\frac{cosn}{n}=0$ =0 2) $\lim_{n\to\infty}\frac{sinn^2}{n}=0$ 3) $\lim_{n\to\infty}nsinn=0$

A) 1 B) 2 C) 3 D) 1,2 E) 2,3

16. Tog'ri tenglikni toping

1) $\lim_{n\to\infty}\frac{n+1}{n}=1$ 2) $\lim_{n\to\infty}\frac{n^2-1}{n^2+1}=1$ 3) $\lim_{n\to\infty}\frac{1-n^3}{1+n^3}=0$

A) 1 B) 2 C) 3 D) 1,2 E) 1, 3

17. Tog'ri tenglikni toping

1) $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a\epsilon R)$ 2) $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$ 3) $\lim\limits_{n\to\infty} \frac{|a|^n}{n!} = 0$

A) 1 B) 2 C) 3 D) 1,2 E) 1, 3

18. $\lim\limits_{n\to\infty} \sqrt{c+\sqrt{c+\cdots+\sqrt{c}}}$ tog'ri tenglikni toping

1)$\sqrt{2c}$ 2) $\sqrt{2+c}$ 3) $\frac{\sqrt{4c+1}+1}{2}$

A) 1 B) 2 C) 3 D) 1,2 E) 1, 3

19. Tog'ri tenglikni toping

$$\lim_{n\to\infty}((1+\frac{1}{1+n})^n : \ e) =?$$

A) 1 B) 2 C)3 D) 1,2 E) 2, 3

20. Quyidagi ketma-katliklarning qaysi biri umumiy hadi $x_n = (-1)^n$ bo'lgan ketma-ketlikning qismiy ketma-ketligi bo'la oladi.

1) -1,-1,-1,…-1,.. 2) 1,1,1,..1,… 3) -1,1,-1,1,…-1,1,…

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

21. Umiy hadi berilgan ketma -ketligidan qaysi hadi fundamental?

1) $x_n = \frac{1}{2n-1}$ 2) $x_n = 1+\frac{1}{3}+\ldots+\frac{1}{2n-1}$ 3) $x_n = 2n-1$

A) 1 B) 2 C) 3 D) 1,2 E) 2, 3

22. Fundamental ketma-ketlik bo'lmagan ketma-ketlikning umumiy hadini toping.

1) $x_n = \frac{1}{n}$ 2) $x_n=\frac{n-1}{n}$ 3) $x_n=\sin n$

A) 1 B) 2 C) 3 D) 1,2 E) 1, 3

23. Qaysi javobda to'g'ri teorema keltirilgan

1) $\{x_n\}, \{y_n\}, \{z_n\}$ ketma-ketliklar barchasi $n \geq n_0$ uchun $\{x_n\} \leq \{y_n\} \leq \{z_n\}$ tengsizlikni qanoatlantirib, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$ bo'lsa, $\{y_n\}$, ketma-ketlik ham yaqinlashuvchi ketma-ketlik va $\lim_{n\to\infty} y_n = a$ bo'ladi.

2) $\{x_n\}, \{y_n\}, \{z_n\}$ ketma-ketliklar barchasi $n \geq n_0$ uchun $\{x_n\} < \{y_n\} < \{z_n\}$ tengsizlikni qanoatlantirib, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$ bo'lsa, $\{y_n\}$, ketma-ketlik ham yaqinlashuvchi ketma-ketlik va $\lim_{n\to\infty} y_n = a$ bo'ladi.

3) $\{x_n\}, \{y_n\}, \{z_n\}$ ketma-ketliklar barchasi $n \geq n_0$ uchun $\{x_n\} < \{y_n\} < \{z_n\}$ tengsizlikni qanoatlantirib, $\lim_{n\to\infty} y_n = a$ bo'lsa, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$ bo'ladi.

A) 1 B) 2 C) 3 D) 1,2 E) 1, 3

5§. FUNKSIYA VA UNING LIMITI

5.1. Funksiya tushunchasi

X va Y haqiqiy sonlarning biror to‘plamlari bo‘lsin.

Ta’rif. Agar X to‘plamdagi har bir x songa biror qoida yoki qonunga ko‘ra Y to‘plamdan bitta y son mos qo‘yilsa, X to‘plamda funksiya berilgan (aniqlangan) deyiladi va $f: x \to y$ yoki $y = f(x)$ ko‘rinishida belgilanadi.

Bunda $X-$ funksiyaning aniqlanish to‘plami (sohasi), $Y-$ funksiyaning o‘zgarish to‘plami (sohasi) deyiladi. Bu yerda x erkli o‘zgaruvchi (funksiya argumenti), y esa ekrsiz o‘zgaruvchi (x o‘zgaruvchining funksiyasi) deyiladi.

Masalan: 1) f - har bir haqiqiy x songa uning butun qismi $[x]$ ni mos qo‘yuvchi qoida bo‘lsin. Demak, $f: x \to [x]$ yoki $y = [x]$

funksiyaga ega bo‘lamiz. Bu funksiyaning aniqlanish to‘plami $X = R,$ o‘zgarish to‘lami esa $Y = Z$ bo‘ladi.

5.2. Funksiyaning berilish usullari

Funksiya ta’rifidagi har bir x ga bitta y ni mos qo‘yadigan qoida yoki qonun turli usullarda berilishi mumkin.

1. Ko‘pincha x va y o‘zgaruvchilar orasidagi bog‘lanish formulalar yordamida ifodalanadi. Bunda argument x ning har bir qiymatiga mos keladigan y funksiyaning qiymatini x ustiga qo‘shish, ayirish, ko‘paytirish, bo‘lish, darajaga ko‘tarish, ildiz chiqarish, logarifmlash va h.k. amallar bajarish natijasida topiladi. Odatda bunday usul funksiyaning ***analitik*** **usulda berilishi** deyiladi.

Misol. x va y o‘zgaruvchilar ushbu $y = \sqrt{1+x^2}$ formula yordamida berilgan bo‘lsin. Bu funksiyaning aniqlanish sohasi $X = \{x : x \in R\}$ to‘plamdan iborat. Bunda har bir x ga mos keladigan y ning qiymati avvalo x ni kvadratga ko‘tarish, so‘ngra unga 1 qo‘shish va yig‘indidan kvadrat ildiz chiqarish kabi amallarni bajarish natijasida topiladi.

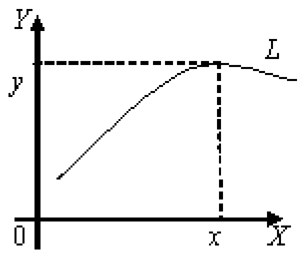
2. Ba’zi hollarda $x\,(x \in X)$ va $y\,(y \in Y)$ o‘zgaruvchilar orasidagi bog‘lanish jadval orqali berilgan bo‘lishi mumkin. Masalan ish davomida ishchining ishlab chiqargan mahsulotini kuzatganimizda, t_1 vaqtda T_1 birlik mahsulot tayyorlagan, t_2 vaqtda T_2 birlik mahsulot tayyorlagan va hokazo bo‘lsin. Natijada quyidagi jadval hosil bo‘ladi:

Vaqt, t	t_1	t_2	t_3	$\ldots$	t_k
Mahsulot birligi, T	T_1	T_2	T_3	$\ldots$	T_k

Bu jadvalda t vaqt bilan mahsulot ishlab chiqarish miqdori T orasidagi funksional bog‘lanishni ifodalaydi, bunda t-argument, T esa funksiya bo‘ladi. Bog‘lanishning bunday berilishi funksiyaning ***jadval usulda berilishi*** deyiladi.

3. XOY tekisligida shunday L chiziq berilgan bo‘lsinki, OX o‘qida joylashgan nuqtalardan shu o‘qqa o‘tkazilgan perpendikulyar bu L chiziqni faqat bitta nuqtada kesib o‘tsin.

OX o‘qida bunday nuqtalardan iborat to‘plamni X orqali belgilaylik. X to‘plamdan ixtiyoriy x ni olib, bu nuqtadan OX o‘qiga perpendikulyar o‘tkazamiz. Bu perpendikulyarning L chiziq bilan kesishgan nuqtasining ordinatasini y bilan belgilaymiz va olingan x ga bu y ni mos qo‘yamiz (10-rasm). Natijada X to‘plamdan olingan har bir x ga yuqorida ko‘rsatilgan qoidaga ko‘ra bitta y mos qo‘yilib, funksiya hosil bo‘ladi. Bunda x va y o‘zgaruvchilar orasidagi bog‘lanish L chiziq yordamida berilgan bo‘ladi. Odatda f ning bunday berilishi uning ***grafik usulda berilishi*** deyiladi.



10-rasm

3. Funksiyalar so‘zlar orqali ham berilishi mumkin.

5.3. Funksiyaning xususiy qiymati

Biror X to‘plamda $y = f(x)$ funksiya aniqlangan bo‘lsin. $x_0 \in X$ ga mos keluvchi y_0 miqdor $y = f(x)$ funksiyaning $x = x_0$ nuqtadagi xususiy qiymati deyiladi va $f(x_0) = y_0$ kabi belgilanadi.

5.4. Funksiyaning grafigi

Tekislikning $(x, f(x))$ nuqtalaridan iborat ushbu

$$(x,\ f(x)) = \{(x,\ f(x)) : x \in X,\ f(x) \in Y\}$$

to‘plam $y = f(x)$ funksiyaning grafigi deyiladi.

5.5. Juft va toq funksiya

Ta’rif. Agar ixtiyoriy $x \in X$ uchun $f(-x) = f(x)$ bo‘lsa, $f(x)$ funksiya ***juft funksiya***, $f(-x) = -f(x)$ bo‘lsa, $f(x)$ funksiya ***toq funksiya***

deyiladi.

Misol. $y = x^2, y = |x|$, $x \in R$ funksiyalar uchun $(-x)^2 = x^2, |-x| = x$ bo‘lgani sababli ular juft funksiyalardir. $y = tgx, y = x^3$, $x \in R$ funksiyalar uchun $tg(-x) = -tgx,\ (-x)^3 = -x^3$ bo‘lganligi sababli ular toq funksiyalardir.

Juft funksiya grafigi Oy ga nisbatan toq funksiya esa koordinata boshiga nisbatan simmetrik bo‘ladi.

5.6. Davriy funksiya va funksiyaning davri

Ta’rif. Agar shunday o‘zgarmas T ($T \neq 0$) son mavjud bo‘lib, $\forall x \in X$ uchun

$$x + T \in X, \quad x - T \in X,$$

$$f(x+T) = f(x)$$

bo‘lsa, $f(x)$ funksiya davriy funksiya deyiladi va bu shartlarni qanoatlantiruvchi musbat T larning eng kichigi (agar u mavjud bo‘lsa) funksiyaning davri deyiladi.

5.7. Eng kichik musbat davr

Agar $T \neq 0$ va ixtiyoriy $x \in X$ uchun $f(x+T) = f(x)$ munosabat o‘rinli bo‘lsa, bu munosabat ixtiyoriy kT $(k = \pm 1, \pm 2, ...)$ uchun ham o‘rinli bo‘ladi.

Demak, $\pm T, \pm 2T, \ldots$ lar ham $f(x)$ funksiyaning davrlari bo‘ladi. $f(x)$ funksiyaning musbat davrlar to‘plamini M deb belgilaylik. Agar $T_0 = \inf M$ ham $f(x)$ funksiyaning davri bo‘lsa, ya’ni $T_0 \in M$ bo‘lsa, u ***eng kichik musbat davr*** (asosiy davr) deyiladi. Eng kichik musbat davr mavjud bo‘lishi ham mumkin, mavjud bo‘lmasligi ham mumkin.

5.8. Davriy funksiyaning xossalari

1°. Agar X to‘plamda berilgan $f(x)$ va $g(x)$ funksiyalarning har biri davriy funksiyalar bo‘lib, $T \neq 0$ ularning davri bo‘lsa, u holda $f(x) \pm g(x)$, $f(x) \cdot g(x)$ va $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ funksiyalar ham davriy funksiyalar bo‘ladi va T ularning ham davri bo‘ladi.

2°. X to‘plamda berilgan $f(x)$ funksiya davriy funksiya, $T \neq 0$ uning davri bo‘lsin. g esa $f(x)$ ning qiymatlari to‘plami $\{f(x) : x \in X\}$ da berilgan ixtiyoriy funksiya bo‘lsin. U holda $g(f(x))$ murakkab funksiya ham davriy funksiya bo‘ladi va uning davri T bo‘ladi.

Misol. $f(x) = \log_2 \cos x$. Bu funksiya argumenti $\cos x$ funksiya eng kichik musbat davri 2π bo‘lganidan $f(x)$ funksiya eng kichik musbat davri ham 2π bo‘ladi.

3°. $f(x)$ davriy funksiya, $T\neq 0$ soni uning davri bo‘lsin. Agar $x_0\in X$ bo‘lsa, u holda barcha x_0+kT ko‘rinishidagi $(k=0,\pm 1,\pm 2,...)$ nuqtalar ham shu sohaga tegishli bo‘ladi: $x_0+kT\in X$ $(k=0,\pm 1,\pm 2,...)$.

Agar x_0 nuqta $f(x)$ funksiyaning berilish sohasiga tegishli bo‘lmasa ($x\overline{\in}X$), u holda barcha x_0+kT ko‘rinishidagi $(k=0,\pm 1,\pm 2,...)$ nuqtalar ham shu sohaga tegishli bo‘lmaydi ($x_0+kT\overline{\in}X$).

4°. Agar $f(x)$ davriy funksiya bo‘lsa, bu funksiya o‘zining har bir qiymatini x argumentning cheksiz ko‘p qiymatlarida qabul qilinadi.

Natija. Agar $f(x)$ davriy funksiya bo‘lsa, berilish sohasida monoton funksiya bo‘lmaydi.

Natija. Agar $f(x)$ funksiya davriy funksiya bo‘lsa, u holda ixtiyoriy $a\in R$ uchun $f(x)=a$ tenglama yechimga ega bo‘lmaydi yoki cheksiz ko‘p yechimga ega bo‘ladi.

Misol. $f(x)=x^2-1$ davriymas, chunki $x^2-1=0$ $(a=0)$ ikkita yechimga ega.

5°. $f(x)$ davriy funksiya bo‘lsin. Agarda

$$f(x+T)=f(x)$$

T ga nisbatan tenglama sifatida qaralsa (x ni esa parametr deyilsa), u holda $f(x+T)=f(x)$ tenglama x parametrning barcha qiymatlari uchun umumiy bo‘lgan noldan farqli kamida bitta $T=T_1$ yechimga ega bo‘ladi.

Bu xossaga ko‘ra, $f(x)$ funksiyaning davriymasligini ko‘rsatish uchun x ning ikkita $x=x_0$, $x=x_1$ qiymatlarida T ga nisbatan ushbu $f(x_0+T)=f(x_0)$, $f(x_1+T)=f(x_1)$ tenglamalarning noldan farqli umumiy yechimga ega emasligini ko‘rsatish yetarli.

6°. $f(x)$ davriy funksiya bo‘lib, $T\neq 0$ uning davri bo‘lsin. Agar uzunligi T ga teng bo‘lgan biror $[\alpha,\alpha+T]$ oraliqda $|f(x)|\leq M$ $(x\in[\alpha,\alpha+T])$ bo‘lsa, argument x ning ixtiyoriy qiymatida ham shu tengsizlik o‘rinli bo‘ladi.

5.9. To‘plamda o‘suvchi funksiya

$f(x)$ funksiya X to‘plamda berilgan bo‘lsin.

Ta’rif. Agar x argumentning X to‘plamdan olingan ixtiyoriy x_1 va x_2 qiymatlari uchun $x_1<x_2$ bo‘lishidan $f(x_1)\leq f(x_2)$ $(f(x_1)<f(x_2))$ tengsizlik kelib chiqsa, $f(x)$ funksiya X to‘plamda ***o‘suvchi (qat’iy o‘suvchi)*** deyiladi.

Misol. $f(x)=\sqrt{x^3}$ funksiya $X=\{x: x\geq 0\}$ da qat'iy o'suvchi. Darhaqiqat, ixtiyoriy $x_1\in X$, $x_2\in X$ nuqtalar olib, $x_1<x_2$ bo'lsin deb qaraylik. U holda

$$f(x_2)-f(x_1)=\sqrt{x_2^3}-\sqrt{x_1^3}>0,$$

chunki katta sondan katta ildiz chiqadi.
Demak, $x_1<x_2$ tengsizlik bajarilganda $f(x_1)<f(x_2)$ tengsizlik ham bajariladi.

5.10. To'plamda kamayuvchi funksiya

$f(x)$ funksiya X to'plamda berilgan bo'lsin.

Ta'rif. Agar x argumentning X to'plamdan olingan ixtiyoriy x_1 va x_2 qiymatlari uchun $x_1<x_2$ bo'lishidan $f(x_1)\geq f(x_2)$ $(f(x_1)>f(x_2))$ tengsizlik kelib chiqsa, $f(x)$ funksiya X to'plamda ***kamayuvchi (qat'iy kamayuvchi)*** deyiladi.

5.11. Monoton fuksiya

O'suvchi va kamayuvchi funksiyalar umumiy nom bilan monoton funksiyalar deyiladi.

5.12. Teskari funksiya

$y=f(x)$ funksiya X to'plamda aniqlangan bo'lib, Y esa funksiya qiymatlaridan iborat to'plam bo'lsin: $Y=\{f(x): x\in X\}$. Shu bilan birga Y to'plamdan olingan har bir y ga X to'plamdan faqat bitta x mos kelsin. Bu holda Y to'plamdan olingan har bir y ga X to'plamdan faqat bitta x mos qo'yilishini ifodalaydigan funksiyaga keladi. Bu funksiya $y=f(x)$ ga nisbatan ***teskari funksiya*** deyiladi va u $x=f^{-1}(y)$ ko'rinishida belgilanadi.

5.13. Murakkab funksiya

$y=f(x)$ funksiya X sohada aniqlangan bo'lib, $Y_f=\{f(x): x\in X\}$ esa funksiya qiymatlaridan iborat to'plam bo'lsin. So'ngra Y_f to'plamda o'z navbatida biror $z=\varphi(y)$ funksiya berilgan bo'lsin. Natijada X to'plamdan olingan har bir x ga Y_f to'plamdan bitta y $(f: x\to y)$ son va Y_f to'plamdan olingan bunday y songa bitta z $(\varphi: y\to z)$ son mos qo'yiladi: $x\overset{f}{\to}y\overset{\varphi}{\to}z$. Demak, X to'plamdan olingan har bir x ga bitta z son mos qo'yiladi.

Odatda, bunday holda f va φ funksiyalarning ***murakkab funksiyasi*** beriladi va u $z=\varphi(f(x))$ kabi belgilanadi.

Misol. $z=\ln\sin x$ funksiyani qaraylik. Bu funksiya $z=\ln y,\ y=\sin x$ funksiya yordamida hosil bo‘lgan funksiya. $y=\sin x$ funksiya $R=(-\infty,+\infty)$ aniqlangan bo‘lib, $z=\ln y$ funksiya esa $y>0$, ya’ni $\sin x>0$ da mavjud. Demak, $z=\ln\sin x$ murakkab funksiya $X=\{x: x\in(2k\pi;\,(2k+1)\pi),\quad k=0,\pm1,\pm2,...\}$ to‘plamda aniqlangan.

5.14. Elementar funksiyalar

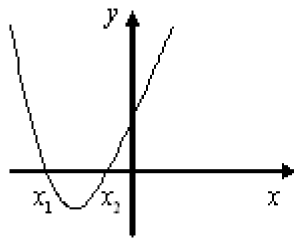
1. Butun ratsional funksiyalar

$$y=a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n$$

ko‘rinishidagi funksiya (bunda $n\in N$ va $a_0, a_1, ..., a_{n-1}, a_n$-o‘zgarmas sonlar) ***butun ratsional funksiya*** deyiladi. Butun ratsional funksiya ko‘phad deb ham yuritiladi. Bu funksiya $R=(-\infty,+\infty)$ da aniqlangan.

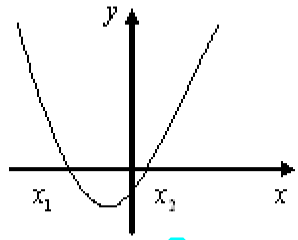
Misol. $y=ax+b$ chiziqli funksiya va $y=ax^2+bx+c$ kvadrat uchhadlar butun ratsional funsiyalardir. Ma’lumki, chiziqli funksiyaning grafigi tekislikda to‘g‘ri chiziqni, kvadrat uchhadning grafigi esa ***parabolani*** ifodalaydi. Kvadrat uchhad grafigining holati a koeffisient hamda diskriminant $D=b^2-4ac$ ning ishoralariga bog‘liq bo‘ladi.

1) $a>0,\ D>0,\ x_1<0,\ x_2<0$



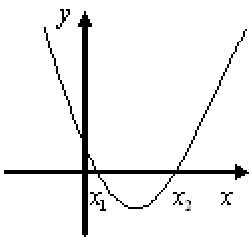
Grafigi I,II,III choraklarda.

2) $a>0,\ D>0,\ x_1<0,\ x_2>0$



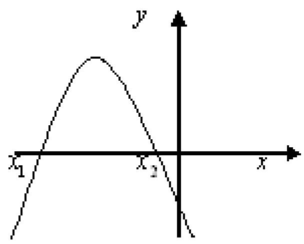
Grafigi I,II,III,IV choraklarda.

3) $a>0,\ D>0,\ x_1>0,\ x_2>0$



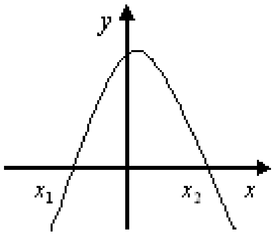
Grafigi I,II,IV choraklarda.

4) $a<0,\ D>0,\ x_1<0,\ x_2<0$.



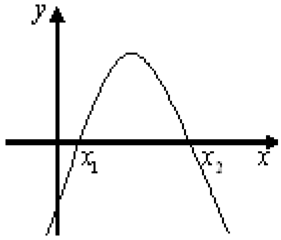
Grafigi II,III,IV choraklarda.

5) $a<0,\ D>0,\ x_1<0,\ x_2>0$.



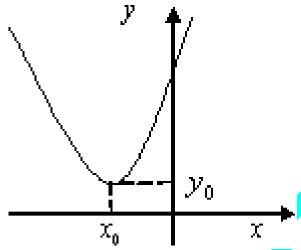
Grafigi I,II,III,IV choraklarda.

6) $a<0,\ D>0,\ x_1>0,\ x_2>0$



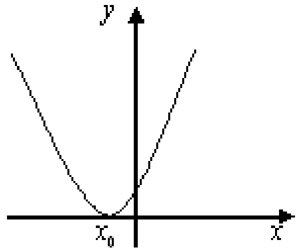
Grafigi I,III,IV choraklarda.

7) $a>0,\ D<0,\ x_0=-\dfrac{b}{2a},\ y_0=\dfrac{4ac-b^2}{4a}>0$.



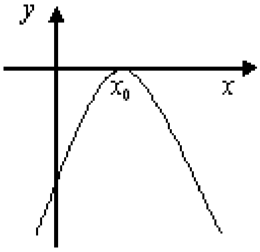
Grafigi I,II choraklarda.

8) $a>0,\ D=0,\ y=0$.



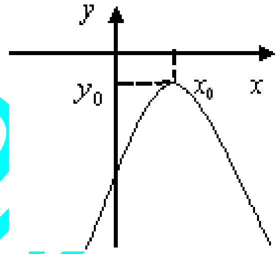
Grafigi I,II choraklarda.

9) $a<0,\ D=0,\ y=0$.



Grafigi III,IV choraklarda.

10) $a<0,\ D<0,\ x_0=-\dfrac{b}{2a},\ y_0=\dfrac{4ac-b^2}{4a}<0$



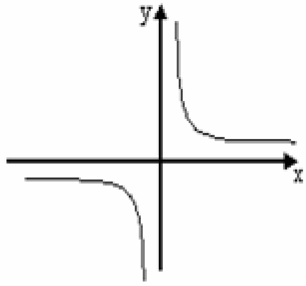
Grafigi III,IV choraklarda

2. Kasr ratsional funksiyalar

Ikki butun ratsional funksiyaning nisbatidan tuzilgan

$$y=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+a_m}$$

funksiya kasr ***ratsional funksiya*** deyiladi. Kasr ratsional funksiya $x=R\setminus\{x:x\in R,\ b_0x^m+b_1x^{m-1}+\ldots+b_{m-1}x+a_m=0\}$ to'plamda,



11-rasm

ya'ni maxrajni nolga aylantiruvchi nuqtalardan farqli barcha haqiqiy sonlardan iborat to'plamda aniqlangan.

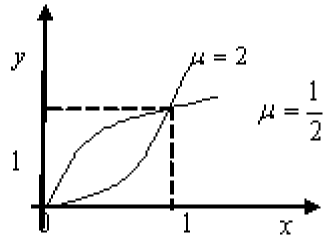
Misol. $y=\dfrac{1}{x}$ va $y=\dfrac{ax+b}{cx+d}$ lar kasr ratsional funksiyalar bo'ladi.

Ma'lumki, $y=\frac{1}{x}$ funksiyaning grafigi teng yonli giperboloiddan iborat (11-rasm).

3. Darajali funksiya

Ushbu $y=x^{\mu}$ ko'rinishidagi funksiyaga darajali funksiya deyiladi. Bunda μ ixtiyoriy o'zgarmas haqiqiy musbat son. Darajali funksiyaning aniqlanish sohasi μ ga bog'liq. μ butun son bo'lganda ratsional funksiyaga ega bo'lamiz.

Agar μ ratsional, masalan $\mu=\frac{1}{m}>0$ bo'lsa, m juft bo'lganda $x^{\mu}=x^{\frac{1}{m}}$ funksiyaning aniqlanish sohasi $X=[0,+\infty)$, toq bo'lganda esa funksiyaning aniqlanish sohasi $R=(-\infty,+\infty)$ oraliqdan iborat bo'ladi. μ irratsional son bo'lganda $x>0$ deb olinadi.

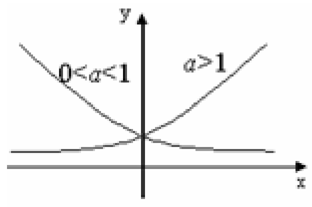


12-rasm

Darajali funksiyaning grafigi har doim tekislikning $(0,0)$ hamda $(1,1)$ nuqtalaridan o'tadi (12-rasm).

4. Ko'rsatkichli funksiya

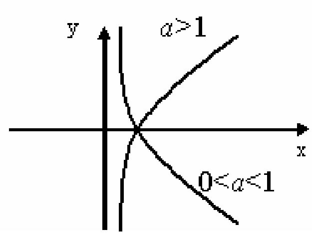
$y=a^{x}$ ko'rinishidagi funksiya ***ko'rsatkichli funksiya*** deyiladi, bunda $a>0$ va $a\neq 1$. Ko'rsatkichli funksiyaning aniqlanish sohasi R to'lamdan iborat bo'lib, funksiya qiymatlari esa har doim musbat bo'ladi. Bu funksiyaning grafigi OX o'qidan yuqorida joylashgan va doim tekislikning $(0,1)$ nuqtasidan o'tadi (13-rasm).



13-rasm

5. Logarifmik funksiya

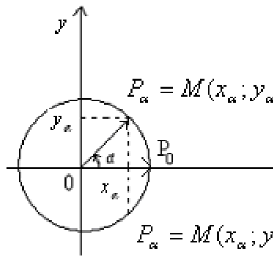
$y=\log_{a}x$ ko'rinishidagi funksiyaga ***logarifmik funksiya*** deyiladi, bunda $a>0$ va $a\neq 1$. Logarifmik funksiya $X=(0,+\infty)$ intervalda aniqlangan. Bu funksiyaning grafigi OY o'qini o'ng tomonida joylashgan va doim tekislikning (1, 0) nuqtasidan o'tadi (14-rasm).



14-rasm

6. Trigonomertik funksiyalar

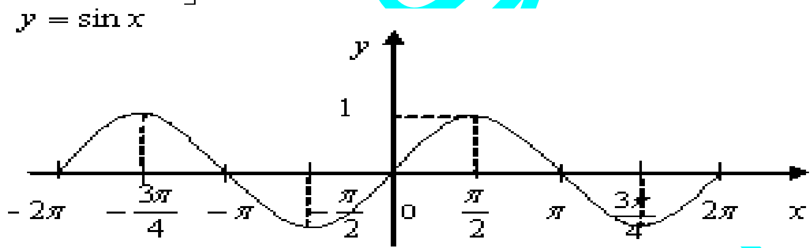
Markazi koordinatalar boshida bo'lgan, radiusi 1 ga teng aylana berilgan bo'lsin (trigonometrik aylana). Koordinata boshi atrofida OP_0 ($|OP| = 1$) vektorni soat strelkasi harakatiga teskari yo'nalishda biror burchakka burish $P_0 = M(1;0)$ nuqtani $P_\alpha = M(x_\alpha; y_\alpha)$ nuqtaga o'tkazsin. Bu musbat yo'nalishli burilish hisoblanadi. Soat strelkasi harakati bo'ylab α burchakka burish manfiy burchakka burish hisoblanadi. Bu holda α o'zgarishi bilan P_α nuqtaning koordinatalari x_α va y_α lar ham turlicha o'zgaradi (15-rasm).



15-rasm

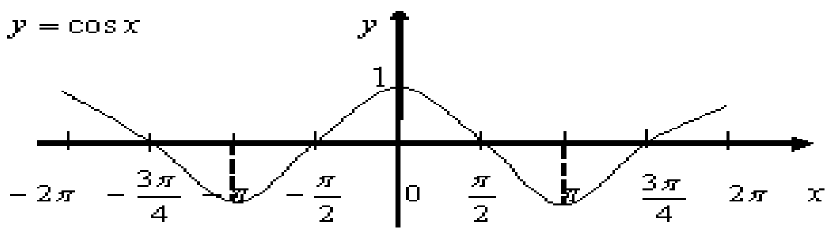
$OP_0 = (1;0)$ vektorni α burchakka burish bilan hosil qilingan $OP_\alpha = (x_\alpha; y_\alpha)$ vektorning absissasi α burchakning kosinusi, ordinatasi esa uning sinusi deb aytiladi va mos ravishda $x_\alpha = \cos\alpha$, $y_\alpha = \sin\alpha$ deb belgilanadi ($tg = \frac{\sin x}{\cos x}$, $ctg = \frac{\cos x}{\sin x}$).

1. $y = \sin x$ funksiyaning aniqlanish sohasi barcha haqiqiy sonlar to'plamidan iborat: $D(y) = R$. Qiymatlar sohasi esa $[-1, 1]$ kesmadan iborat: $E(y) = [-1, 1]$. Ushbu funksiya $\left[-\frac{\pi}{2} + 2\pi n; \frac{\pi}{2} + 2\pi n\right]$, ($n \in Z$) oraliqda o'suvchi; $\left[\frac{\pi}{2} + 2\pi n; \frac{3\pi}{2} + 2\pi n\right]$, ($n \in Z$) oraliqda kamayuvchidir (16-rasm).



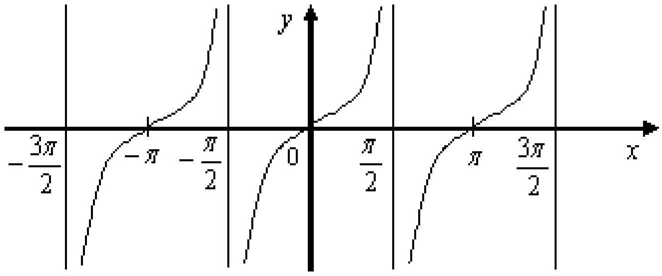
16-rasm

2. $y = \cos x$ funksiyaning aniqlanish sohasi barcha haqiqiy sonlar to'plamidan iborat: $D(y) = R$. Qiymatlar sohasi esa $[-1, 1]$ kesmadan iborat: $E(y) = [-1, 1]$. Ushbu funksiya $[2\pi n, \pi + 2\pi n]$,($n \in Z$) oraliqda kamayuvchi, $[\pi + 2\pi n; 2\pi + 2\pi n]$, ($n \in Z$) oraliqda o'suvchidir (17-rasm).



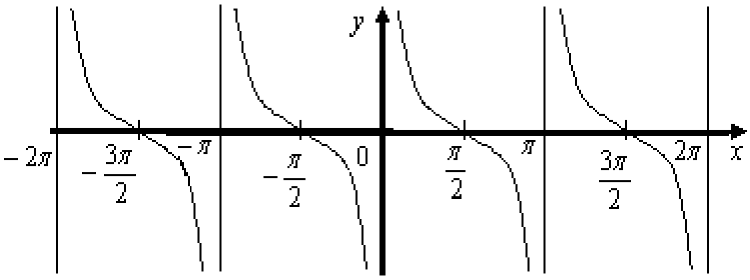
17-rasm

3. $y = tgx$ funksiyaning aniqlanish sohasi $D(x) = \left(-\frac{\pi}{2} + \pi n; \frac{\pi}{2} + \pi n\right)$, ($n \in Z$); qiymatlar sohasi esa $E(y) = R$. $y = tgx$ funksiya har bir $\left(-\frac{\pi}{2} + \pi n; \frac{\pi}{2} + \pi n\right)$, ($n \in Z$) oraliqda o‘suvchi (18-rasm).



18-rasm

4. $y = ctgx$ funksiyaning aniqlanish sohasi $D(y) = (\pi n; \pi + \pi n)$, ($n \in Z$); qiymatlar sohasi esa $E(y) = R$. $y = ctgx$ funksiya har bir $(\pi n; \pi + \pi n)$, ($n \in Z$) oraliqda kamayuvchi (19-rasm).



19-rasm

7. Giperbolik funksiyalar

Ushbu $y = e^x$ ko‘rsatkichli funksiya yordamida tuzilgan quyidagi

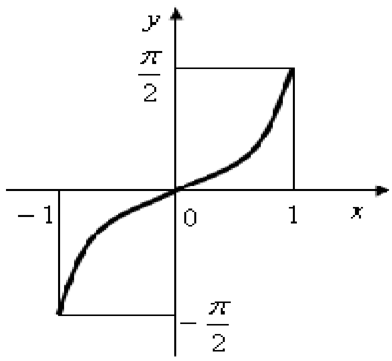
$$shx = \frac{e^x - e^{-x}}{2},\ chx = \frac{e^{-x} + e^x}{2},\ thx = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ cthx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

funksiyalar ***giperbolik*** (mos ravishda giperbolik sinus, giperbolik kosinus, giperbolik tangens, giperbolik kotangens) ***funksiyalar*** deyiladi.

shx, chx, thx funksiyalar R da, $cthx$ funksiya esa $X = R \setminus \{0\}$ to‘plamda aniqlangan. Quyidagilar o‘rinli $thx = \frac{shx}{chx}$,

$cthx = \frac{chx}{shx}$, $sh2x = 2shx \cdot chx$, $ch^2x - sh^2x = 1$,

$sh^2x + ch^2x = ch2x$.



8. Teskari trigonometrik funksiyalar

$y = \arcsin x$ funksiya

1) $D(y) = [-1, 1]$;

2) $E(y)=\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$;

3) $\arcsin(-x)=-\arcsin x$ tenglik bajarilgani uchun bu funksiya toqdir;

4) $(0;1]$ oraliqda musbat, $[-1;0)$ oraliqda manfiy qiymatlidir; 20-rasm

5) $[-1;1]$ kesmada o‘suvchi bo‘lib, bu oraliqning chap oxirida o‘zining eng kichik $-\frac{\pi}{2}$ qiymatiga, o‘ng oxirida eng katta $\frac{\pi}{2}$ qiymatiga erishadi.

Bu funksiyaning grafigi $y=\sin x$ funksiyanig $\left[\frac{-\pi}{2};\frac{\pi}{2}\right]$ oraliqdagi grafigini $y=x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi (20-rasm).

$y=\arccos x$ funksiya

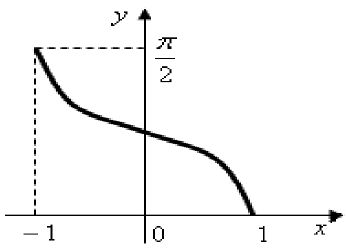
1) $D(y)=[-1,1]$;

2) $E(y)=[0;\pi]$;

3) Funksiya toq ham, juft ham emas;

4) $[-1;1]$ oraliqda musbat qiymatlidir;

5) $[-1;1]$ kesmada kamayuvchi bo‘lib, bu oraliqning chap oxirida o‘zining eng katta π qiymatiga, o‘ng oxirida eng kichik 0 qiymatiga erishadi.



21-rasm

Grafigi $y=\cos x$ funksiyanig $[0;\pi]$ oraliqdagi grafigini $y=x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi (21-rasm).

$y=arctgx$ funksiya

1) $D(y)=R$;

2) $E(y)=\left(\frac{-\pi}{2};\frac{\pi}{2}\right)$;

3) $arctg(-x)=-arctgx$ tenglik bajarilgani uchun bu funksiya toqdir;

4) R_+ oraliqda musbat, R_- oraliqda manfiy qiymatlidir;

5) R haqiqiy sonlar to‘plamida o‘suvchi.

Grafigi $y=tgx$ funksiyanig $\left(\frac{-\pi}{2};\frac{\pi}{2}\right)$ oraliqdagi grafigini $y=x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi (22-rasm).

$y=arcctgx$ funksiya

1) $D(x)=R$;

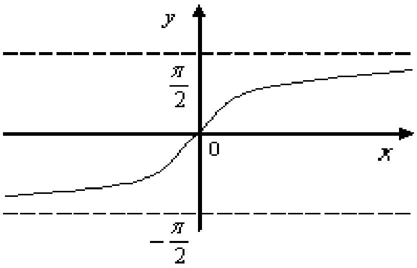
2) $E(y)=(0;\pi)$;

3) Funksiya toq ham, juft ham emas;

4) R to‘plamning har bir nuqtasida musbat qiymatlidir;
5) R to‘plamda kamayuvchi.

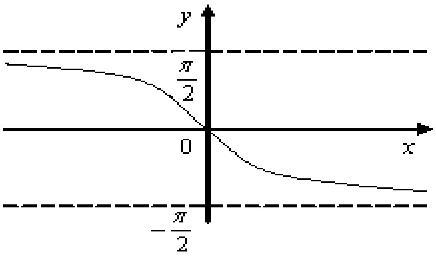
Grafigi $y = ctgx$ funksiyanig $(0;\pi)$ oraliqdagi grafigini $y = x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi (23-rasm).

$y = arctgx$



22-rasm

$y = arcctgx$



23-rasm

5.15. Funksiyaning yuqoridan (quyidan) chegaralanganligi

$y = f(x)$ funksiya X to‘plamda aniqlangan bo‘lsin.

Ta’rif. Agar shunday o‘zgarmas M (o‘zgarmas m) son topilsaki, ixtiyoriy $x \in X$ uchun

$$f(x) \le M \quad (f(x) \ge m)$$

bo‘lsa, $f(x)$ funksiya X to‘plamda ***yuqoridan (quyidan) chegaralangan*** deyiladi. Agar $f(x)$ funksiya ham yuqoridan, ham quyidan chegaralangan bo‘lsa, ya’ni shunday o‘zgarmas M va m sonlar topilsaki, ixtiyoriy $x \in X$ uchun

$$m \le f(x) \le M$$

bo‘lsa, $f(x)$ funksiya X to‘plamda ***chegaralangan*** deyiladi.

Ta’rif. Agar ixtiyoriy M (ixtiyoriy m) son olinganda ham, shunday $x_0 \in X$ ($x_0' \in X$) son topilsaki,

$$f(x_0) > M \quad (f(x_0') < m)$$

bo‘lsa, $f(x)$ funksiya X to‘plamda ***yuqoridan (quyidan) chegaralanmagan*** deyiladi.

5.16. To‘plamning limit nuqtasi

Ta’rif. Agar a nuqtaning har bir atrofida X to‘plamning a dan farqli kamida bitta nuqtasi bo‘lsa, a nuqta X ***to‘plamning limit nuqtasi*** deyiladi.

Ta’rif. Agar a nuqtaning har bir o‘ng (chap) atrofida X to‘plamning a dan farqli kamida bitta nuqtasi bo‘lsa, a nuqta X ning ***o‘ng (chap) limit nuqtasi*** deyiladi (a cheksiz ham bo‘lishi mumkin).

5.17. To'plam limit nuqtasining xossalari

1°. X to'plamning limit nuqtasi shu to'plamga tegishli bo'lishi ham, tegishli bo'lmasligi ham mumkin.

2°. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, a nuqtaning har bir atrofida X to'plamning cheksiz ko'p nuqtalari bo'ladi.

3°. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, X to'plam nuqtalaridan a ga intiluvchi $\{x_n\}$, ($x_n \in X, x_n \neq a, n = 1, 2, 3, ...$) ketma-ketlik tuzish mumkin.

5.18. Funksiyaning limiti

$X = \{x\}$ haqiqiy sonlar to'plami berilgan bo'lib, a nuqta uning limit nuqtasi bo'lsin. Bu to'plamda aniqlangan $y = f(x)$ funksiyani qaraymiz.

Ta'rif (Geyne ta'rifi). Agar X to'plamning nuqtalaridan tuzilgan a ga intiluvchi har qanday $\{x_n\}$ ($x_n \neq 0, n = 1,2,...$) ketma-ketlik olinganda ham mos $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b (chekli yoki cheksiz) limitga intilsa, shu b ga $f(x)$ funksiyaning a ***nuqtadagi (yoki*** $x \to a$ ***dagi) limiti*** deyiladi va u $\lim_{x \to a} f(x) = b$ yoki $x \to a$ da $f(x) \to b$ ko'rinishida belgilanadi.

Ta'rif (Koshi ta'rifi). Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, argument x ning $0 < |x - a| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning a ***nuqtadagi*** ($x \to a$ dagi) ***limiti*** deyiladi.

5.19. Funksiyaning o'ng (chap) limiti

Ta'rif (Geyne ta'rifi). Agar X to'plamning nuqtalaridan tuzilgan va har bir hadi a dan katta (kichik) bo'lib a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham mos $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b ga intilsa, shu b ni $f(x)$ funksiyaning a nuqtadagi ***o'ng (chap) limiti*** deyiladi va $\lim_{x \to a+0} f(x) = b$ yoki $f(a+0) = b$ ($\lim_{x \to a-0} f(x) = b$ yoki $f(a-0) = b$) kabi belgilanadi.

Ta'rif (Koshi ta'rifi). Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta = \delta(\varepsilon)$ son topilsaki, argument x ning $a < x < a + \delta$ $(a - \delta < x < a)$ tengsizliklarni qanoatlantiruvchi barcha $x \in X$ qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning a nuqtadagi ***o'ng (chap) limiti*** deyiladi.

Chap va o‘ng limitlar ***bir tomonli limitlar*** deyiladi.

Funksiyaning o‘ng (chap) limitlari quyidagi ko‘rinishda belgilanadi:

$$\lim_{x\to a+0} f(x)=b \text{ yoki } f(a+0)=b,$$

$$\left(\lim_{x\to a-0} f(x)=b \text{ yoki } f(a-0)=b\right).$$

Misol. Ushbu

$$f(x)=\begin{cases}-2x+1, & agar \ \ x<1\\ x+1, & agar \ \ x>1\end{cases}$$

funksiyaning $x=1$ nuqtadagi o‘ng va chap limitlarini toping.

$$\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}(x+1)=2;$$

$$\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}(-2x+1)=-1.$$

Teorema (*limitga ega bo‘lish haqida*). $f(x)$ funksiya a nuqtada b limitga ega bo‘lishi uchun uning shu nuqtadagi o‘ng va chap limitlari mavjud bo‘lib,

$$f(a+0)=f(a-0)=b$$

tenglik o‘rinli bo‘lishi zarur va yetarli.

5.20. Chekli limitga ega bo‘lgan funksiyaning xossalari

X to‘plam berilgan bo‘lib, a nuqta uning limit nuqtasi bo‘lsin va shu to‘plamda aniqlangan $f(x)$ funksiyani qaraymiz.

1°. Agar ushbu $\lim\limits_{x\to a} f(x)=b$ limit mavjud bo‘lib, $b>p$ $(b<q)$ bo‘lsa, a nuqtaning yetarli kichik atrofidan olingan x $(x\neq a)$ ning qiymatlarida $f(x)>p$ $(f(x)<q)$ bo‘ladi.

2°. Agar ushbu $\lim\limits_{x\to a} f(x)=b$ limit mavjud bo‘lsa, a nuqtaning yetarli kichik atrofidan olingan x $(x\neq a)$ ning qiymatlarida $f(x)$ funksiya chegaralangan bo‘ladi.

3°. Agar x argumentning a nuqtaning biror $\dot{U}_\delta(a)$ atrofidan olingan barcha qiymatlarida $f_1(x)\leq f_2(x)$ tengsizlik o‘rinli bo‘lib, $\lim\limits_{x\to a} f_1(x)$, $\lim\limits_{x\to a} f_2(x)$ limitlar mavjud bo‘lsa, u holda $\lim\limits_{x\to a} f_1(x)\leq \lim\limits_{x\to a} f_2(x)$ tengsizlik o‘rinli bo‘ladi.

4°. Agar x argumentning a nuqtaning biror $\dot{U}_\delta(a)$ atrofidan olingan barcha qiymatlarida $f_1(x)\leq f(x)\leq f_2(x)$ tengsizlik o‘rinli bo‘lsa va

$\lim\limits_{x\to a} f_1(x)$, $\lim\limits_{x\to a} f_2(x)$ limitlar mavjud bo‘lib, $\lim\limits_{x\to a} f_1(x)=\lim\limits_{x\to a} f_2(x)=b$ bo‘lsa, u holda $\lim\limits_{x\to a} f(x)=b$ bo‘ladi.

5.21. Chekli limitga ega bo‘lgan funksiyalar ustida arifmetik amallar

X to‘plam berilgan bo‘lib, a nuqta uning limit nuqtasi bo‘lsin. Bu to‘plamda $f(x)$ va $g(x)$ funksiyalar aniqlangan.

1°. Agar $x\to a$ da $f(x)$ va $g(x)$ funksiyalar limitga ega bo‘lsa, $f(x)\pm g(x)$ funksiya ham limitga ega va

$$\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

tenglik o‘rinli.

2°. Agar $x\to a$ da $f(x)$ va $g(x)$ funksiyalar limitga ega bo‘lsa, $f(x)\cdot g(x)$ funksiya ham limitga ega va

$$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

tenglik o‘rinli.

Natija. Agar $x\to a$ da $f(x)$ funksiya limitga ega bo‘lsa, unda $k\cdot f(x)$ ($k=const$) funksiya ham limitga ega va

$$\lim_{x\to a}[k\cdot f(x)]=k\cdot\lim_{x\to a} f(x)$$

tenglik o‘rinli.

3°. Agar $x\to a$ da $f(x)$ va $g(x)$ funksiyalar limitga ega bo‘lib, $\lim\limits_{x\to a} g(x)\neq 0$ bo‘lsa, $\dfrac{f(x)}{g(x)}$ ($g(x)\neq 0$) funksiya ham limitga ega va

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

tenglik o‘rinli.

Teorema. Agar 1) $\lim\limits_{x\to a}\varphi(x)=c$ bo‘lsa, a nuqtaning shunday $U_\delta(a)$ atrofi mavjud bo‘lib, barcha $x\in U_\delta(a)$ lar uchun $\varphi(x)\neq c$ bo‘lsa, 2) c nuqta T to‘plamning limit nuqtasi bo‘lib, $\lim\limits_{t\to c} f(t)=b$ bo‘lsa, u holda $x\to a$ da murakkab funksiya $y=f(\varphi(x))$ ham limitga ega va

$$\lim_{x\to a} f(\varphi(x))=b$$

bo‘ladi.

Teorema. $f(x)$ funksiya X to‘plamda o‘suvchi bo‘lib, u yuqoridan chegaralangan bo‘lsa, funksiya a nuqtada chekli limitga ega, yuqoridan chegaralanmagan bo‘lsa, uning limiti $+\infty$ bo‘ladi.

Teorema. Agar $f(x)$ funksiya X to‘plamda kamayuvchi bo‘lib, u quyidan chegaralangan bo‘lsa, funksiya a nuqtada chekli limitga ega, quyidan chegaralanmagan bo‘lsa, uning limiti $-\infty$ bo‘ladi.

Ta’rif (*Koshi sharti (kriteriysi)*). Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, x argumentning $0 < |x' - a| < \delta$, $0 < |x'' - a| < \delta$ tengsizliklarni qanoatlantiruvchi x' va x'' qiymatlarida

$$|f(x'') - f(x')| < \varepsilon$$

tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya uchun a ***nuqtada Koshi sharti*** bajarilgan deyiladi.

Teorema (*Koshi teoremasi*). $f(x)$ funksiya a nuqtada chekli limitga ega bo‘lishi uchun bu funksiya uchun a nuqtada Koshi sharti bajarilishi zarur va yetarli.

5.22. Murakkab funksiyaning limiti

Biror X to‘plamda aniqlangan $t = \varphi(x)$ funksiya va uning qiymatlaridan iborat T to‘plamda $y = f(t)$ funksiya aniqlangan bo‘lib, ular yordamida murakkab funksiya $y = f(\varphi(x))$ hosil qilingan bo‘lsin. Bu murakkab funksiya X to‘plamda aniqlangan. Shu bilan birga a son X to‘plamning limit nuqtasi bo‘lsin.

Teorema. Agar 1) $\lim\limits_{x \to a} \varphi(x) = c$ bo‘lib, a nuqtaning shunday $U_\delta(a)$ atrofi mavjud bo‘lsaki, barcha $x \in U_\delta(a)$ lar uchun $\varphi(x) \neq c$ bo‘lsa, 2) c nuqta T to‘plamning limit nuqtasi bo‘lib, $\lim\limits_{t \to c} f(t) = b$ bo‘lsa, u holda $x \to a$ da murakkab funksiya $y = f(\varphi(x))$ ham limitga ega va $\lim\limits_{x \to a} f(\varphi(x)) = b$ bo‘ladi.

5.23. Cheksiz katta va kichik funksyalar

Biror X to‘plam berilgan bo‘lib, a uning limit nuqtasi bo‘lsin. Bu to‘plamda $\alpha(x)$, $\beta(x)$ funksiyalar aniqlangan bo‘lsin.

Ta’rif. Agar $x \to a$ da $\alpha(x)$ funksiyaning limiti nolga teng bo‘lsa, $\alpha(x)$ funksiya $x \to a$ da ***cheksiz kichik funksiya*** deyiladi.

Misol $f(x) = x^2$ funksiya $x \to 0$ da cheksiz kichik funksiya bo‘ladi, chunki $\lim\limits_{x \to 0} x^2 = 0$.

Ta'rif. Agar $x \to a$ da $\beta(x)$ funksiyaning limiti ∞ bo'lsa, $\beta(x)$ funksiya $x \to a$ da ***cheksiz katta funksiya*** deyiladi.

Misol. $f(x) = \frac{1}{x}$ funksiya $x \to 0$ da cheksiz katta funksiya bo'ladi.

Agar $f(x)$ funksiya $x \to a$ da chekli b limitga ega bo'lsa ($\lim\limits_{x \to a} f(x) = b$), u holda $\alpha(x) = f(x) - b$ funksiya $x \to a$ da ***cheksiz kichik funksiya*** bo'ladi.

5.24. Cheksiz katta (kichik) funksyalarning xossalari

1°. Chekli sondagi cheksiz kichik funksiyalar yig'indisi cheksiz kichik funksiya bo'ladi.

2°. Chegaralangan funksiyaning cheksiz kichik funksiya bilan ko'paytmasi cheksiz kichik funksiya bo'ladi.

3°. Agar $\alpha(x)$ ($\alpha(x) \neq 0$) cheksiz kichik funksiya bo'lsa, $\frac{1}{\alpha(x)}$ cheksiz katta funksiya bo'ladi.

4°. Agar $\beta(x)$ cheksiz katta funksiya bo'lsa, $\frac{1}{\beta(x)}$ cheksiz kichik funksiya bo'ladi.

5.25. Funksiyalarni taqqoslash

$X \subset R$ to'plamda $f(x)$ va $g(x)$ funksiyalar aniqlangan bo'lsin. Biror a nuqtaning $U_\delta(a)$ ($U_\delta(a) \subset X$) atrofida $f(x)$ va $g(x)$ funksiyalar quyidagicha taqqoslanadi.

Ta'rif (O-belgi). Agar $f(x)$ va $g(x)$ funksiyalar uchun shunday $\delta > 0$ va $C > 0$ sonlar topilsaki, barcha $x \in U_\delta(a)$ lar uchun

$$|f(x)| \leq C|g(x)|$$

bo'lsa, u holda $x \to a$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan ***chegaralangan*** deyiladi va $f(x) = O(g(x))$ kabi belgilanadi.
Agar $f(x) = O(g(x))$ va $g(x) = O(f(x))$ bo'lsa, $f(x)$ funksiya $g(x)$ funksiyalar $x \to a$ da ***bir xil tartibli funksiyalar*** deyiladi.

Ta'rif (ekvivalentlik, ~). Agar

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1$$

bo'lsa, $x \to a$ da $g(x)$ va $f(x)$lar ***ekvivalent funksiyalar*** deyiladi va $f(x) \sim g(x)$ kabi belgilanadi.

Ta'rif (o-belgi). Agar $g(x)$ va $f(x)$ funksiyalar uchun

$$f(x)=\alpha(x)\cdot g(x)$$

bo'lib, bunda $\lim\limits_{x\to a}\alpha(x)=\lim\limits_{x\to a}g(x)=0$ bo'lsa, u holda $x\to a$ da $f(x)$ funksiya $g(x)$ ga nisbatan ***yuqori tartibli cheksiz kichik funksiya*** deyiladi va $f(x)=o(g(x))$ kabi belgilanadi.

Misol. Ushbu $|x|^3=o(x^2)$ munosabat $x\to 0$ da o'rinli. $|x|^3=|x|\cdot x^2$ tenglikdan va $\lim\limits_{x\to 0}|x|=0$ ekanligidan $|x|^3=o(x^2)$ o'rinli.

5.26. Limitlarni hisoblashda kerak bo'ladigan ajoyib va muhim limitlar

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1 \quad \left(\lim\limits_{x\to 0}\dfrac{x}{\sin x}=1\right)$.

2. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e \quad \left(\lim\limits_{x\to\infty}\left(1+\dfrac{\alpha}{x}\right)^x=e^{\alpha}\right)$.

3. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$.

4. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a \quad (a>0)$. **4′.** $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

5. $\lim\limits_{x\to 0}\dfrac{(1+x)^{\alpha}-1}{x}=\alpha$.

5′. Agar $P(x)=a_1x+a_2x^2+...+a_nx^m$, m -butun son, $\lim\limits_{x\to 0}\dfrac{\sqrt{1+P(x)}-1}{x}=\dfrac{a_1}{m}$.

6. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1 \quad \left(\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e\right)$.

7. $\lim\limits_{x\to +0}x^a\ln x=\lim\limits_{x\to +\infty}x^{-a}\ln x=\lim\limits_{x\to +\infty}x^a e^{-x}=0 \quad (a>0)$.

5.27. Funksiyaning uzluksizligi

$X\subset R$ to'plamda $f(x)$ funksiya aniqlangan bo'lib, x_0 ($x_0\in X$) to'plamning limit nuqtasi bo'lsin.

Ta'rif. Agar $x\to a$ da $f(x)$ funksiya limiti mavjud va $\lim\limits_{x\to 0}f(x)=f(a)$ bo'lsa, $f(x)$ ***funksiya*** a ***nuqtada uzluksiz*** deyiladi.

Ta'rif (*Koshi ta'rifi*). Ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta>0$ son topilsaki, funksiya argumenti $x\in X$ ning $|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$|f(x)-f(a)|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya a nuqtada uzluksiz deyiladi.

Ta'rif *(Geyni ta'rifi).* Agar X to'plamning elementlaridan tuzilgan va a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham funksiya qiymatlaridan tuzilgan mos $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona $f(a)$ ga intilsa, berilgan funksiya a ***nuqtada uzluksiz*** deyiladi.

Agar

$$\lim_{x\to x_0} f(x) = f(x_0)$$

munosabat o'rinli bo'lsa, ushbu

$$\lim_{x-x_0\to 0}[f(x) - f(x_0)] = 0$$

munosabat ham o'rinli bo'ladi. Odatda $x - x_0$ ayirma ***argument orttirmasi***, $f(x) - f(x_0)$ esa ***funksiyaning*** x_0 ***nuqtadagi orttirmasi*** deyiladi.

Ular mos ravishda Δx va Δy ($\Delta f(x_0)$) kabi belgilanadi:

$$\Delta x = x - x_0, \ \Delta y = \Delta f(x_0) = f(x) - f(x_0).$$

Demak, $x = x_0 + \Delta x$, $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

Natijada $\lim\limits_{x\to x_0} f(x) = f(x_0)$ munosabat

$$\lim_{\Delta x\to 0}\Delta y = \lim_{\Delta x\to 0}\Delta f(x_0) = 0$$

ko'rinishiga ega bo'ladi.

5.28. Uzluksiz funksiyaning xossalari

Funksiyalar uzluksizligi nuqtada va oraliqda qaraladi.

1°. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, u holda a nuqtaning yetarli kichik atrofida funksiya chegaralangan bo'ladi.

2°. Agar $f(x)$ funksiya a nuqtada uzluksiz va $f(a) \neq 0$ bo'lsa, u holda a ning yetarli kichik atrofidan olingan barcha x nuqtalarda funksiya qiymatlarining ishorasi $f(a)$ ning ishorasi bilan bir xil bo'ladi (agar har xil ishora bo'lsa $f(a) = 0$ bo'ladi).

3°. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, u holda ixtiyoriy $\varepsilon > 0$ uchun a nuqtaning shunday kichik atrofi topiladiki, bu atrofdan olingan ixtiyoriy x', x'' nuqtalar uchun

$$|f(x') - f(x'')| < \varepsilon$$

o'rinli bo'ladi.

Teorema *(Bolsano-Koshining birinchi teoremasi).* Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo'lib, segmentning chetki nuqtalarida har xil ishorali qiymatlarga ega bo'lsa, u holda shunday c ($a<c<b$) nuqta topiladiki, u nuqtada funksiya nolga aylanadi: $f(c)=0$.

Teorema *(Bolsano-Koshining ikkinchi teoremasi).* Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lib, uning chetki nuqtalarida $f(a)=A$, $f(b)=B$ qiymatlarga ega va $A\neq B$ bo‘lsa, A va B orasida har qanday C son olinganda ham a bilan b orasida shunday c nuqta topiladiki,

$$f(c)=C$$

bo‘ladi.

Teorema *(Veyrshtrassning birinchi teoremasi).* Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsa, u shu segmentda chegaralangan bo‘ladi.

Teorema *(Veyrshtrassning ikkinchi teoremasi).* Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsa, funksiya shu segmentda o‘zining aniq yuqori hamda aniq quyi chegaralariga erishadi.

5.29. Tekari funksiyaning mavjudligi

Agar $f(x)$ funksiya X oraliqda aniqlangan, uzluksiz va qat’iy o‘suvchi (qat’iy kamayuvchi) bo‘lsa, bu funksiya qiymatlaridan iborat $Y=\{f(x):x\in X\}$ oraliqda teskari $f^{-1}(y)$ funksiya mavjud bo‘lib, u uzluksiz va qat’iy o‘suvchi (qat’iy kamayuvchi) bo‘ladi.

5.30. Funksiyaning tekis uzluksizligi. Kantor teoremasi

Biror $y=f(x)$ funksiya X to‘plamda berilgan bo‘lsin.

Ta’rif. Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta(\varepsilon)>0$ son topilsaki, X to‘plamning $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy x' va x'' ($x',x''\in X$) nuqtalarida

$$|f(x'')-f(x')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya X to‘plamda ***tekis uzluksiz*** deyiladi.

Ta’rif. Shunday musbat ε soni mavjud bo‘lib, ixtiyoriy $\delta>0$ son olinganda ham $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi $x',x''\in X$ nuqtalar topilsaki

$$|f(x')-f(x'')|\geq\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya X to‘plamda ***tekis uzluksiz emas*** deyiladi.

Teorema *(Kantor teoremasi).* Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsa, u shu segmentda tekis uzluksiz bo‘ladi.

5.31. Funksiyaning bir tomonli uzluksizligi

$X \subset R$ da $f(x)$ funksiya aniqlangan bo‘lib, $a \in X$ esa X to‘plamning o‘ng (chap) limit nuqtasi bo‘lsin.

Ta’rif. Agar $x \to a+0$ ($x \to a-0$) da $f(x)$ funksiyaning o‘ng (chap) limiti mavjud va $\lim\limits_{x\to a+0} f(x) = f(a)$ $\left(\lim\limits_{x\to a-0} f(x) = f(a)\right)$ bo‘lsa, $f(x)$ a ***nuqtada o‘ngdan (chapdan) uzluksiz*** deyiladi.

Ta’rif. Agar $f(x)$ funksiya a nuqtada ham o‘ngdan ham chapdan bir vaqtda uzluksiz bo‘lsa, funksiya shu ***nuqtada uzluksiz*** bo‘ladi.

Ta’rif. Agar $f(x)$ funksiya $X \subset R$ to‘plamning har bir nuqtasida uzluksiz bo‘lsa, funksiya X ***to‘plamda uzluksiz*** deyiladi.

5.32. Funksiyaning uzilishi. Uzilish turlari. Funksiya sakrashi

$f(x)$ funksiya X to‘plamda aniqlangan bo‘lib, $a \in X$ nuqta X ***to‘plamning limit nuqtasi*** bo‘lsin.

Ta’rif. Agar $x \to a$ bo‘lganda $f(x)$ funksiyaning limiti mavjud, chekli bo‘lib

$$\lim_{x\to a} f(x) = b \neq f(a) \text{ yoki } \lim_{x\to a} f(x) = \infty$$

$(-\infty, +\infty)$ bo‘lsa yoki mavjud bo‘lmasa, unda $f(x)$ funksiya $x = a$ ***nuqtada uzilishga ega*** deyiladi.

Uzilish turlari: I, II tur va “Bartaraf qilinishi mumkin bo‘lgan” uzilishlar .

1. <u>I-tur uzilish</u>: agar $x \to a$ da $f(x)$ funksiyaning o‘ng ($f(a+0)$) va chap ($f(a-0)$) limitlari mavjud va chekli bo‘lib,

$$f(a-0) \neq f(a+0)$$

bo‘lsa, $f(x)$ funksiya $x = a$ da ***I tur uzilishga ega*** deyiladi.

Misol. $f(x) = \dfrac{\sin x}{|x|}$ funksiya $x = 0$ nuqtada I tur uzilishga ega.

Haqiqatdan,

$$f(O+0) = f(+0) = \lim_{x\to+0} \frac{\sin x}{|x|} = +1 \text{ va } f(O-0) = f(-0) = \lim_{x\to-0} \frac{\sin x}{-x} = -1$$

$f(+0) \neq f(-0)$.

2. <u>Bartaraf qilish mumkin bo‘lgan uzilish</u>: $f(x)$ funksiyaning o‘ng va chap limitlari mavjud va chekli bo‘lib,

$$f(a-0)=f(a+0)\neq f(a)$$

munosabat o‘rinli bo‘lsa, $f(x)$ funksiya $x=a$ nuqtada ***bartaraf qilish mumkin bo‘lgan uzilishga ega*** deyiladi.

3. <u>II tur uzilish:</u> $x\to a$ da $f(x)$ funksiyaning limiti mavjud bo‘lmaydigan boshqa hamma hollarda funksiya a nuqtada ikkinchi tur uzilishga ega deyiladi. Agar chap yoki o‘ng tomonli limitlar $+\infty, -\infty$ bo‘lgan hollar ham II turga kiritiladi.

5.33. Monoton funksiyaning uzluksizligi va uzilishi

$f(x)$ funksiya X oraliqda aniqlangan bo‘lsin.

Teorema. Agar $f(x)$ funksiya X oraliqda monoton funksiya bo‘lsa, u shu oraliqning istalgan nuqtasida yo uzluksiz bo‘ladi, yoki faqat birinchi tur uzilishga ega bo‘ladi.

Teorema. Agar $f(x)$ funksiya X oraliqda monoton bo‘lib, uning qiymatlar to‘plami $Y_f=\{f(x):x\in X\}$ biror oraliqdan iborat bo‘lsa, u holda bu funksiya X da uzluksiz bo‘ladi.

5.34. Uzluksiz funksiyalar ustida arifmetik amallar

$f(x)$ va $g(x)$ funksiyalar X oraliqda berilgan bo‘lsin.

Teorema. Agar $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo‘lsa, ularning yig‘indisi ham a nuqtada uzluksiz bo‘ladi.

Teorema. Agar $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo‘lsa, ularning ko‘paytmasi ham uzluksiz bo‘ladi.

Teorema. Agar $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo‘lsa, ularning $\frac{f(x)}{g(x)}$ $(g(x)\neq 0)$ nisbati ham a nuqtada uzluksiz bo‘ladi.

5.35. Murakkab funksiyaning uzluksizligi

$y=f(x)$ funksiya X to‘plamda, $z=\varphi(y)$ esa Y to‘plamda aniqlangan bo‘lib, $z=\varphi(f(x))$ murakkab funksiya berilgan bo‘lsin.

Teorema. Agar $y=f(x)$ funksiya $a\in X$ nuqtada, $z=\varphi(y)$ funksiya esa a nuqtadaga mos kelgan $y_a=f(a)$ nuqtada uzluksiz bo‘lsa, $z=\varphi(f(x))$ murakkab funksiya a nuqtada uzluksiz bo‘ladi.

5.36. Limitlarni hisoblashda funksiyaning uzluksizligidan foydalanish

$z=\varphi(f(x))$ murakkab funksiya berilgan bo‘lsin.

Agar $\lim\limits_{x\to a} f(x) = f(a) = y_a$ mavjud bo'lib, $z = \varphi(y)$ funksiya y_a nuqtada uzluksiz bo'lsa, u holda $\lim\limits_{x\to a} \varphi(f(x))$ mavjud va

$$\lim_{x\to a}\varphi(f(x)) = \varphi\left(\lim_{x\to a} f(x)\right) = \varphi(y_a)$$

o'rinli.

Agar $\lim\limits_{x\to a} f(x) = b\ (b > 0), \lim\limits_{x\to a} g(x) = c$ o'rinli bo'lsa,

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim\limits_{x\to a} g(x)\ln\left[\lim\limits_{x\to a} f(x)\right]} = e^{c\ln b} = b^c.$$

$f(x)^{g(x)}$ ***funksiya darajali-ko'rsatkichli funksiya*** deyiladi:

1. $\lim\limits_{x\to a} f(x) = 1,\quad \lim\limits_{x\to a} g(x) = \infty$ bo'lsa, 1^∞ aniqmaslik bo'ladi.
2. $\lim\limits_{x\to a} f(x) = 0,\quad \lim\limits_{x\to a} g(x) = 0$ bo'lsa, 0^0 aniqmaslik bo'ladi.
3. $\lim\limits_{x\to a} f(x) = +\infty,\quad \lim\limits_{x\to a} g(x) = 0$ bo'lsa, ∞^0 aniqmaslik bo'ladi.

1^∞ aniqmaslik quyidagicha ochiladi

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a}\left\{\left[1 + (f(x)-1)\right]^{\frac{1}{f(x)-1}}\right\}^{(f(x)-1)g(x)} = e^{\lim\limits_{x\to a}(f(x)-1)g(x)}.$$

Misol. $\lim\limits_{x\to 0}(1+x^2)^{ctg^2 x},\quad f(x) = 1 + x^2,\quad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(1+x^2) = 1,\quad g(x) = ctg^2 x,$

$\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} ctg^2 x = \infty$,. $\lim\limits_{x\to 0}(1+x^2)^{ctg^2 x} = \lim\limits_{x\to 0} e^{(1+x^2-1)ctg^2 x} = e^{\lim\limits_{x\to 0}\left(\frac{x}{tgx}\right)^2} = e.$

M3. Funksiya va uning limitiga doir mashqlar

1. Agar $f(x) = x^2 - 3x + 1$ bo'lsa, $f(0)$, $f(1)$, $f(-2)$, $f(a+2)$, $f(a^2+2)$ hisoblansin.

2. Agar $\varphi = \dfrac{5x+3}{x^2+4}$ bo'lsa, $\varphi(-1)$, $\varphi(2)$, $\varphi(a^2)$, $\varphi\left(\dfrac{5}{x}\right)$, $\varphi\left(\dfrac{1}{x+1}\right)$, $\dfrac{1}{\varphi(x)+\varphi(1)}$ hisoblansin.

3. Agar $f(x) = 2x^3$ bo'lsa, $\dfrac{f(2)-f(3)}{f(1)-f(-1)}$, $f\left(\dfrac{a+b}{3}\right) - f\left(\dfrac{a-b}{3}\right)$ hisoblansin.

4. Agar $f(x) = 2x^2$, $\varphi(x) = 3x^3$ bo'lsa, $\dfrac{f(b)-f(a)}{\varphi(b)-\varphi(a)}$ hisoblansin.

5. Agar $f(x,y) = 2x^3 - 5xy + y^2$ bo'lsa, $f(5{,}2)$ va $f(2{,}5)$ hisoblansin.

6. Quyidagi funksiyalarning qaysilari toq, qaysilari juft ekanligi aniqlansin:

1) $f_1(x) = \dfrac{\sin 5x}{x}$;

2) $f_2(x) = \dfrac{a^x - 3}{a^x + 3}$;

3) $f_3(x) = \dfrac{1}{a^x} - a^x$;

4) $f_4(x) = x\sin^3 x - 5x^3$;

5) $f_5(x)=x^4-2x^2$; 6) $f_6(x)=x^2\cos^2 x+5$;
7) $f_7(x)=\sin x+x^3-5$; 8) $f_8(x)=|x-2|+|x+2|$;
9) $f_9(x)=\lg\dfrac{1+x}{1-x}$; 10) $f_{10}(x)=\sqrt[3]{(x+1)^2}+\sqrt[3]{(x-1)^2}$

7. Quyidagi funksiyalarning aniqlanish sohalari topilsin:

1) $y=\sqrt{x-4}$ 2) $y=\sqrt{|x|+1}$
3) $y=\sqrt{|x|-4}$ 4) $y=\dfrac{\sin x}{\sqrt{5x-x^2}}$
5) $y=\sqrt{-x-1}$ 6) $y=\sqrt[3]{3+\sqrt{x-1}}$
7) $y=-\sqrt{\sin 2x}$ 8) $y=\arcsin\dfrac{x-1}{3}$
9) $y=\log_x(x^2-1)$ 10) $y=\dfrac{x-1}{x^2-4}+\dfrac{x}{x+1}+\dfrac{5}{x-3}$
11) $y=\log_2(x+1)+3\log_{x^2-1}(x+3)$ 12) $y=2^{\sqrt{x-1}}+3^{\sqrt[3]{x-1}}$

8. Quyidagi funksiyalarda y ni x orqali ifodalang.

1) $y=z^2$, $z=x^2+1$ 2) $y=u^2+u$, $u=\sqrt{z+1}$, $z=2^x$
3) $y=\sqrt{z+2}$, $z=tg^2x$ 4) $y=\cos z$, $z=\sqrt{u+1}$, $u=x^2$

9. Quyidagi funksiyalardan $f(g(x))$ va $g(f(x))$ murakkab funksiyalarni tuzing va ularni aniqlanish sohasini toping.

1) $f(x)=x^2$, $g(x)=\sqrt{x}$ 2) $f(x)=2^x$, $g(x)=\log_2 x$
3) $f(x)=x^5$, $g(x)=x+5$ 4) $f(x)=\ln x^2$, $g(x)=\cos x$
5) $f(x)=e^x$, $f(x)=\ln x$

10. Quyidagi munosabatlardan $f(x)$ ni toping.

1) $f(x+2)=x^2+3x+2$ 2) $f\left(\dfrac{1}{x}\right)=x+\sqrt{2+x^2}$
3) $f\left(\dfrac{x}{x+1}\right)=x^2+1$ 4) $f\left(\dfrac{x-1}{x+1}\right)=x^2$, $(x\neq -1)$

11. $y=f(x)$ funksiyaga teskari funksiya $x=\varphi(x)$ ni va aniqlanish sohasini toping.

1) $y=5x+4$, $(-\infty<x<+\infty)$ 2) $y=5x^2$, $(-\infty<x\leq 0)$
3) $y=\sqrt{1-x^2}$, $(-1\leq x\leq 0)$ 4) $y=x^2-1$, $\left(-\infty<x<-\dfrac{1}{2}\right)$
5) $y=\sin x$, $\left(-\dfrac{\pi}{2}\leq x\leq\dfrac{\pi}{2}\right)$ 6) $y=\cos^2 x$, $\left(\pi\leq x\leq\dfrac{3\pi}{2}\right)$
7) $y=10^{x+1}$ 8) $y=\log_x 10$
9) $y=\dfrac{3^x}{1+3^x}$ 10) $y=\dfrac{10^x-10^{-x}}{10^x+10^{-x}}+1$

12. Quyidagi limitlarni toping.

1) $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$

2) $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1}$

3) $\lim\limits_{x\to \infty}\dfrac{5x^2-x+1}{15x^2+x+1}$

4) $\lim\limits_{x\to -2}\dfrac{x^3+3x^2+2x}{x^2-x-6}$

5) $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^2}\right)$

6) $\lim\limits_{x\to 1}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}$, $(m,\ n\in N)$

7) $\lim\limits_{x\to 5}\dfrac{\sqrt{6-x}-1}{3-\sqrt{4+x}}$

8) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x^2}$

9) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x^2}-1}{x^2}$

10) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

11) $\lim\limits_{x\to +\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$

12) $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{a^2-a^2}}$ $(a>0)$

13) $\lim\limits_{x\to +\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

14) $\lim\limits_{x\to 0}\dfrac{\sin\sqrt{2}x}{5x}$

15) $\lim\limits_{x\to 0}\dfrac{\sin 12x}{\sin 6x}$

16) $\lim\limits_{x\to 0}\dfrac{tgnx}{\sin mx}$, $(n,\ m\in N)$

17) $\lim\limits_{x\to 0}\dfrac{5\arcsin 2x}{7x}$

18) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$

19) $\lim\limits_{x\to 0}\dfrac{tgx-\sin x}{x^3}$

20) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$

21) $\lim\limits_{x\to 0}\dfrac{\cos(a+x)-\cos(a-x)}{x}$

22) $\lim\limits_{\alpha\to \beta}\dfrac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$

23) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\cos x}{\frac{\pi}{2}-x}$

24) $\lim\limits_{x\to \pi}\dfrac{\sin x}{\pi^2-x^2}$

25) $\lim\limits_{x\to 0}\left(\dfrac{2+x}{3-x}\right)^x$

26) $\lim\limits_{x\to 1}\left(\dfrac{x-1}{x^2-1}\right)^{x+1}$

27) $\lim\limits_{x\to \infty}\left(\dfrac{x}{x+1}\right)^x$

28) $\lim\limits_{x\to \infty}\left(\dfrac{x-1}{x+3}\right)^{x+2}$

29) $\lim\limits_{x\to \infty}\left(\dfrac{1}{x^2}\right)^{\frac{2x}{x+1}}$

30) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

31) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}$

32) $\lim\limits_{x\to \infty}\left(1+\dfrac{2}{x}\right)^x$

33) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$

13. Ushbu funksiyalarning aniqlanish sohasining har bir nuqtasida uzliksizligini ko'rsating.

1) $y=5x+1$

2) $y=x^2+2$

3) $y=\sqrt{x}$

4) $y=x^3$

5) $y=\dfrac{1}{x}$ 6) $y=\cos x$

7) $y=\dfrac{x}{|x|}$ 8) $y=e^{\frac{1}{x+1}}$

9) $y=\dfrac{x}{\sin x}$ 10) $y=e^{x+\frac{1}{x}}$

14. Funksiyalarning uzilish nuqtalarini va uzilish turini aniqlang.

1) $y=\dfrac{1}{(1+x)^2}$ 2) $y=\dfrac{1+x}{1+x^3}$

3) $y=\dfrac{x^2-1}{x^2-3x+2}$ 4) $y=\dfrac{5}{1+3^{\frac{1}{x}}}$

5) $y=\dfrac{4^{\frac{1}{x}}-1}{4^{\frac{1}{x}}+1}$ 6) $y=e^{x+\frac{1}{x}}$

Testlar

15. $y=\sqrt{4-x^2}$ funksiyaning aniqlanish sohasida nechta natural son bor?

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

16. Juft funksiyalarni aniqlang.

1) $y=x^2|x|$ 2) $y=17^x+17^{-x}$ 3) $y=(x-1)^{15}-(x+1)^{15}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

17. Toq funksiyani aniqlang.

1) $y=|x|$ 2) $y=13^x+13^{-x}$ 3) $y=(x-1)^{20}-(x+1)^{20}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

18. $\lim\limits_{x\to3}\dfrac{x^3-27}{x^2-9}-?$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

19. $\lim\limits_{x\to-3}\dfrac{x^3+27}{x^2-9}-?$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

20. $\lim\limits_{x\to3}\dfrac{x^2-2x+3}{x^2-5x+6}-?$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

21. To'g'ri tenglikni aniqlang.

1) $\lim\limits_{x\to+0}\dfrac{1}{x}=+\infty$ 2) $\lim\limits_{x\to+0}\dfrac{1}{x}=-\infty$ 3) $\lim\limits_{x\to+0}\dfrac{1}{x}=\infty$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

22. To'g'ri tenglikni aniqlang.

1) $\lim\limits_{x\to+0}\log_2 x = +\infty$ 2) $\lim\limits_{x\to+0}\log_{\frac{1}{2}} x = +\infty$ 3) $\lim\limits_{x\to+0}\log_{\frac{1}{2}} x = \infty$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

23. Noto'g'ri tenglikni toping.

1) $x^2 \sim x,\ x\to 0$ 2) $x^3+1\sim x+1,\ x\to -1$ 3) $x-1\sim (x-1)^2,\ x\to 1$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

24. Noto'g'ri tenglikni toping

1) $x^2 = o(x),\ x\to 0$ 2) $x^3+1 = o(x+1),\ x\to -1$ 3) $x-1 = o((x-1)^2),\ x\to 1$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

25. Noto'g'ri tenglikni toping.

1) $x^2 = O(x),\ x\to\infty$ 2) $x^3+1 = O(x+1),\ x\to\infty$ 3) $x-1 = O((x-1)^2),\ x\to\infty$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

26. $x=1$ nuqtada uzluksiz funksiyalarni aniqlang.

1) $f(x) = x^2-2x+1$ 2) $f(x) = \{x\}$ 3) $f(x) = \dfrac{x^2-1}{x-1}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

27. $x=1$ nuqtada birinchi tur uzilishga ega bo'lgan funksiyani aniqlang.

1) $f(x) = x^2-2x+1$ 2) $f(x) = \{x\}$ 3) $f(x) = \dfrac{x^2-1}{x-1}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

28. $f(x) = \{x\}$ funksiyaning $[0;1]$ kesmadagi eng katta qiymatini ko'rsating.

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

29. $f(x) = 2\cos x$ funksiyaning $[-1;1]$ kesmadagi eng katta qiymatini ko'rsating.

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

30. Qaysi funksiyaga R da teskari funksiya mavjud.

1) $y = x$ 2) $y = x^2$ 3) $y = x^3$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

31. To'g'ri tenglikni toping.

1) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} = \dfrac{1}{2}$ 2) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} = 2$ 3) $\lim\limits_{x\to 0}\dfrac{x}{\sin 2x} = 2$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

32. To'g'ri tenglikni toping.

1) $\arcsin(-0{,}5) = \dfrac{7\pi}{6}$ 2) $\arccos(-0{,}5) = \dfrac{2\pi}{3}$ 3) $\arccos(-0{,}5) = \dfrac{\pi}{3}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

33. To'g'ri tenglikni toping.

1) $\lim\limits_{x\to\infty}\left(1-\dfrac{2}{x}\right)^{x}=e^{2}$ 2) $\lim\limits_{x\to 0}(1-2x)^{\frac{1}{x}}=e^{2}$ 3) $\lim\limits_{x\to-\infty}\left(1-\dfrac{2}{x}\right)^{x}=e^{-2}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

34. To'g'ri tenglikni toping.

1) $\lim\limits_{x\to 0}\dfrac{\ln(1-x)}{x}=1$ 2) $\lim\limits_{x\to +0}\dfrac{\ln(1-x)}{x}=1$ 3) $\lim\limits_{x\to -0}\dfrac{\ln(1-x)}{x}=1$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

35. To'g'ri tenglikni toping.

1) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-1}{x}=3$ 2) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}-1}{x}=3$ 3) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}-1}{x}=\dfrac{1}{3}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

36. Qaysi funksiyalar $[-1;1]$ kesmada tekis uzluksiz bo'ladi.

1) $y=x$ 2) $y=\dfrac{1}{x}$ 3) $y=\sin x$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

37. Qaysi funksiyalar $[-1;1]$ kesmada tekis uzluksiz bo'ladi.

1) $y=\sin\dfrac{1}{x}$ 2) $y=\cos\dfrac{1}{x}$ 3) $y=\cos x$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

38. Qaysi funksiyalar $[-1;1]$ kesmada tekis uzluksiz bo'ladi.

1) $y=[x]$ 2) $y=\{x\}$ 3) $y=\dfrac{x}{|x|}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 2, 3

5.37. Funksiya hosilasi va differensiali

$y=f(x)$ funksiya x_0 nuqtaning ($x_0\in R$) biror atrofida berilgan bo'lsin. Bu funksiyaning x_0 nuqtadagi orttirmasi

$$\Delta y=\Delta f(x_0)=f(x_0+x)-f(x_0)$$

ning argument orttirmasi $\Delta x=x-x_0$ ga nisbati

$$\frac{\Delta y}{\Delta x}=\frac{\Delta f(x)}{\Delta x}\quad(\Delta x\neq 0)$$

ni qaraymiz.

Ta'rif. Agar $\Delta x\to 0$ da $\dfrac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

mavjud va chekli bo‘lsa, bu limit $y=f(x)$ funksiyaning x_0 nuqtadagi hosilasi deyiladi va $f'(x)$ yoki $y'_{x=x_0}$ yoki $\left.\frac{dy}{dx}\right|_{x=x_0}$ ko‘rinishlarida belgilanadi.

Misol. $f(x)=\sin x$ funksiyaning $x\in R$ nuqtadagi hosilasini ta’rifdan foydalanib toping.

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\sin\frac{x+\Delta x-x}{2}\cdot\cos\frac{x+\Delta x+x}{2}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{2\sin\frac{\Delta x}{2}\cdot\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}=\cos x$$

Demak, $(\sin x)'=\cos x,\quad \forall x\in R$.

5.38. Bir tomonli hosilalar

Ta’rif. Agar $\Delta x\to+0$ ($\Delta x\to-0$) da $\frac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x\to+0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to+0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$\left(\lim_{\Delta x\to-0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to-0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right)$$

mavjud va chekli bo‘lsa, bu limit $f(x)$ funksiyaning x_0 ***nuqtadagi o‘ng (chap) hosilasi deyiladi*** va uni $f'(x_0+0)$ ($f'(x_0-0)$) ko‘rinishida belgilanadi.

Funksiyaning o‘ng va chap hosilalari ***bir tomonli hosilalar*** deyiladi.

Misol. $f(x)=|x|$ ni qaraylik. Ma’lumki, $\Delta y=|x+\Delta x|-|x|$, $x=0$ da $\Delta y=|\Delta x|$. u holda $\lim\limits_{\Delta x\to+0}\frac{\Delta y}{\Delta x}=1$, $\lim\limits_{\Delta x\to-0}\frac{\Delta y}{\Delta x}=-1$. Demak, $f(x)=|x|$ funksiyaning $x=0$ nuqtadagi o‘ng hosilasi 1 ga, chap hosilasi -1 ga teng. Funksiya $x=0$ nuqtada hosilaga ega emas.

5.39. Cheksiz hosilalar

$y=f(x)$ funksiya x_0 nuqtaning ($x_0\in R$) biror atrofida berilgan bo‘lib, u x_0 nuqtada uzluksiz bo‘lsin.

Ta’rif. Agar $\Delta x\to 0$ da $\frac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$+\infty$ (yoki $-\infty$) bo‘lsa, u ham $f(x)$ funksiyaning x_0 nuqtadagi hosilasi bo‘lib, bunga ***cheksiz hosila*** deyiladi.

Demak,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = +\infty .$$

5.40. Funksiya hosilasining geometrik va mexanik ma’nosi

$f(x)$ funksiya (a,b) oraliqda aniqlangan va uzluksiz bo‘lib, $x_0 \in (a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo‘lsin. U holda $f(x)$ funksiya grafigiga $M_0(x_0, f(x_0))$ nuqtada o‘tkazilgan urinma mavjud. Funksiyaning x_0 nuqtadagi hosilasi $f'(x_0)$ esa bu ***urinmaning burchak koeffisientini*** ifodalaydi (bu hosilaning geometrik ma’nosi). Urinmaning tenglamasi

$$y = f(x_0) + f'(x_0) \cdot (x - x_0)$$

ko‘rinishida bo‘ladi.

Agar $f'(x_0) = \pm\infty$ bo‘lsa, $f(x)$ funksiya grafigiga $(x_0, f(x_0))$ nuqtada o‘tkazilgan urinma Ox o‘qiga perpendikulyar bo‘ladi.

Moddiy nuqtaning to‘g‘ri chiziqli harakati $s = f(t)$ funksiya bilan ifodalangan bo‘lsin, bunda t-vaqt, s shu vaqt ichida o‘tilgan masofa (yo‘l).

$s = f(x)$ funksiyaning t_0 nuqtadagi hosilasi $f'(t_0)$ harakat qilayotgan moddiy nuqtaning t_0 vaqtdagi oniy tezligini bildiradi (bu hosilaning fizik ma’nosi).

5.41. Teskari funksiyaning hosilasi

Agar $y = f(x)$ funksiya x_0 nuqtada $f'(x_0) \neq 0$ hosilaga ega bo‘lsa, bu funksiyaga teskari $x = f^{-1}(y)$ funksiya x_0 nuqtada hosilaga ega va

$$\left(f^{-1}(y)\right)'_{y=y_0} = \frac{1}{f'_x(x_0)}$$

bo‘ladi.

Misol. Ushbu $y = \arccos x$ funksiyaning hosilasini toping.

Ravshanki, $y = \arccos x$ funksiya $x = \cos y$ funksiyaga $(0 < y < \pi)$ teskari funksiyadir. Unda yuqoridagi qoidaga ko‘ra $y' = (\arccos x)' = \dfrac{1}{(\cos y)'}$

bo‘ladi. Ma’lumki, $(\cos y)' = -\sin y$. Demak,
$$y' = (\arccos x)' = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}, \quad (-1<x<1).$$

5.42. Murakkab funksiyaning hosilasi

$u = f(x)$ funksiya (a,b) oraliqda, $y = F(x)$ funksiya esa (c,d) oraliqda berilgan bo‘lib,
$$y = F(f(x))$$
murakkab funksiyani qaraylik.

Agar $u = f(x)$ funksiya $x \in (a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo‘lib, $y = F(u)$ funksiya esa x_0 nuqtaga mos u_0 $(u_0 = f(x_0))$ nuqtada $F'(u_0)$ hosilaga ega bo‘lsa, u holda murakkab funksiya $F(f(x))$ ham x_0 nuqtada hosilaga ega va
$$[F(f(x))]'_{x=x_0} = F'(u_0)\cdot f'(x_0) \quad \left([F(f(x))]' = F'(f(x))\cdot f'(x)\cdot x'\right)$$
bo‘ladi.

Misol. Ushbu $y = \sin x^2$ funksiyaning hosilasini toping.

Ko‘rinib turibdiki, bu murakkab funksiya bo‘lib, uni $y = F(u) = \sin u,\, u = f(x) = x^2$ deb qarash mumkin. Yuqoridagi formulaga ko‘ra: $y' = (\sin x^2)' = = (\sin u)'_{u=x^2}\cdot (x^2)' = \cos x^2 \cdot 2x = 2x\cos x^2$.

5.43. Hosilani hisoblashning sodda qoidalari

$f(x)$ va $g(x)$ funksiyalar $x \in (a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lsin. U holda $f(x) \pm g(x)$, $f(x)\cdot g(x)$, $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ funksiyalar ham hosilaga ega va

1) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$;

2) $[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$, $\quad [(cf(x))' = c\cdot f'(x)]$;

3) $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)} \quad (g(x) \neq 0)$, $\quad \left[\left(\dfrac{A}{f(x)}\right)' = -\dfrac{Af'(x)}{f^2(x)}\right]$

bo‘ladi.

5.44. Hosilalar jadvali

1. $(x^\mu)' = \mu \cdot x^{\mu-1}, \quad (\mu > 0)$.

1′. $(f^\mu(x))' = \mu \cdot f^{\mu-1}(x)\cdot f'(x)$

2. $(a^x)' = a^x \cdot \ln a,\ (a>0,\ a\neq 1)\ \left((e^x)' = e^x\right)$.

2′. $\left(a^{f(x)}\right)' = a^{f(x)} \cdot f'(x) \cdot \ln a$.

2″. $\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$.

3. $(\log_a x)' = \dfrac{1}{x \ln a} \quad (x > 0,\ a > 0,\ a \neq 1)$.

3′. $(\log_a f(x))' = \dfrac{f'(x) \log_a e}{f(x)}$.

3″. $(\ln x)' = \dfrac{1}{x}$; 3‴. $(\ln f(x)) = \dfrac{f'(x)}{f(x)}$.

4. $(\sin x)' = \cos x$.

4′. $(\sin(f(x)))' = \cos f(x) \cdot f'(x)$.

5. $(\cos x)' = -\sin x$.

5′. $(\cos(f(x)))' = -\sin f(x) \cdot f'(x)$

6. $(tgx)' = \dfrac{1}{\cos^2 x},\ x \neq \dfrac{\pi}{2} + k\pi;\ k = 0, \pm 1, \pm 2, \ldots$.

6′. $(tg(f(x)))' = \dfrac{f'(x)}{\cos^2 f(x)}$.

7. $(ctgx)' = -\dfrac{1}{\sin^2 x},\ x \neq k\pi;\ k = 0, \pm 1, \pm 2, \ldots$.

7′. $(ctg(f(x)))' = -\dfrac{f'(x)}{\sin^2 f(x)}$.

8. $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$.

8′. $(\arcsin(f(x)))' = \dfrac{f'(x)}{\sqrt{1 - f^2(x)}}, \quad -1 < f(x) < 1$.

9. $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$.

9′. $(\arccos(f(x)))' = -\dfrac{f'(x)}{\sqrt{1 - f^2(x)}}, \quad -1 < f(x) < 1$.

10. $(arctgx)' = \dfrac{1}{1 + x^2}$.

10′. $(arctg(f(x)))' = \dfrac{1}{1 + f^2(x)} \cdot f'(x)$.

11 . $(arcctgx)' = -\dfrac{1}{1 + x^2}$.

11′. $(arcctg(f(x)))' = -\dfrac{1}{1 + f^2(x)} \cdot f'(x)$.

12. $(shx)' = chx$.

12′. $(sh(f(x))' = chf(x) \cdot f'(x)$

13. $(chx)' = shx$.

13′. $(ch(f(x)))' = shf(x)\cdot f'(x)$.

14. $(thx)' = \dfrac{1}{ch^2 x}$.

14′. $(th(f(x)))' = \dfrac{f'(x)}{ch^2 f(x)}$.

15. $(cthx)' = -\dfrac{1}{sh^2 x}, x \neq 0$.

15′. $cth(f(x))' = -\dfrac{f'(x)}{sh^2 f(x)}$.

5.45. Ayrim funksiyalarning hosilalari

1. $(\sqrt[n]{x^m})' = \left(x^{\frac{m}{n}}\right)' = \dfrac{m}{n}x^{\frac{m}{n}-1}$, $\left((\sqrt[3]{x})' = \left(x^{\frac{1}{3}}\right)' = \dfrac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}\right)$.

2. $\left(F^n(f(x))\right)' = nF^{n-1}(f(x))\cdot F'(f(x))f'(x)$.

3. $\left(\sin^5(\cos 3x)\right) = -5\sin^4(\cos 3x)\cdot\cos(\cos 3x)\cdot\sin 3x\cdot 3$.

4. $(u(x)^{v(x)})' = u(x)^{v(x)}\left[v'(x)\cdot\ln u(x) + \dfrac{v(x)}{u(x)}\cdot u'(x)\right]$.

5. $(x^x)' = x^x(\ln x + 1)$.

6. $\left(\sqrt{f(x)}\right)' = \dfrac{f'(x)}{2\sqrt{f(x)}}$ $\left(f(x)\right)' = \dfrac{f'(x)}{2\sqrt{f(x)}}$, $\left(\sqrt{x} = \dfrac{1}{2\sqrt{x}}\right)$.

5.46. Parametrga bog'liq funksiyaning hosilasi

Agar funksiya quyidagi parametr ko'rinishida berilgan bo'lsa,

$$\left.\begin{array}{l} x = \varphi(t) \\ y = \psi(x) \end{array}\right\} \quad \alpha < t < \beta$$

va $\varphi(t)$ va $\psi(t)$ lar yetarli tartibda hosilaga ega, quyidagi o'rinli:

$$\frac{dy}{dx} = \frac{\psi'(t)}{\varphi'(t)} = \frac{y'_t}{x'_t}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{\psi'}{\varphi'}\right)}{dx} = \frac{d\left(\frac{\psi'}{\varphi'}\right)}{\varphi' dt} = \frac{\psi''\varphi' - \varphi''\psi'}{(\varphi')^3}$$

5.47. Funksiyaning differensiali

$y = f(x)$ funksiya x_0 nuqtaning ($x_0 \in R$) biror atrofida berilgan bo'lsin. Bu funksiyaning x_0 nuqtadagi orttirmasi

$$\Delta y = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

ni qaraylik. Ravshanki, bu orttirma Δx ga bog'liqdir.

Ta'rif. Agar $y = f(x)$ funksiyaning x_0 nuqtadagi orttirmasi Δy ni

$$\Delta y = \Delta f(x_0) = A\cdot\Delta x + \alpha(\Delta x)$$

(bunda A-o‘zgarmas, $\alpha = \alpha(\Delta x)$ bo‘lib, $\Delta x \to 0$ da $\alpha(\Delta x) \to 0$) ko‘rinish da ifodalash mumkin bo‘lsa, funksiya x_0 nuqtada ***differensiallanuvchi*** deyiladi.

$\Delta y = \Delta f(x_0) = A\cdot\Delta x + \alpha(\Delta x)$ munosabatni quyidagicha

$$\Delta y = \Delta f(x_0) = A\cdot\Delta x + \theta(\Delta x)$$

ko‘rinishida yozish mumkin.

Ta’rif. $f(x)$ funksiya orttirmasi Δy ning Δx ga nisbati chiziqli bosh qismi $A\cdot\Delta x = f'(x)$ $(x\in(a,b))$ berilgan $f(x)$ funksiyaning x ***nuqtadagi differensiali*** deyiladi va dy yoki $df(x)$ kabi belgilanadi

$$dy = df(x) = f'(x)\Delta x.$$

Agar $f(x) = x$ bo‘lsa, $dy = df(x) == dx = \Delta x$. Demak,

$$dy = f'(x)dx.$$

5.48. Funksiya differensialining geometrik ma’nosi

$f(x)$ funksiyaning x nuqtadagi grafigiga $(x, f(x))$ nuqtada o‘tkazilgan urinma orttirmasini (dy) ifodalaydi.

5.49. Funksiyaning differensiallanuvchi bo‘lish sharti

Teorema. $f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo‘lishi uchun uning shu nuqtada chekli $f'(x)$ hosilaga ega bo‘lishi zarur va yetarli.

5.50. Elementar funksiyalarning differensiali jadvali

1. $d(x^\mu) = \mu x^{\mu-1}dx \quad (x>0)$.
2. $d(a^x) = a^x\cdot\ln a\cdot dx \quad (a>0,\ a\neq 1)$.
3. $d(\log_a x) = \dfrac{1}{x}\log_a e\cdot dx \quad (x>0,\ a>0,\ a\neq 1);\ d(\ln x) = \dfrac{dx}{x}$.
4. $d(\sin x) = \cos x\cdot dx$.
5. $d(\cos x) = -\sin x\cdot dx$.
6. $d(tgx) = \dfrac{1}{\cos^2 x}dx \quad \left(x\neq \dfrac{\pi}{2}+k\pi;\ k=0;\pm1,\pm2,\ldots\right)$.
7. $d(ctgx) - \dfrac{1}{\sin^2 x}dx \quad (x\neq k\pi;\ k=0;\pm1,\pm2,\ldots)$.
8. $d(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}dx \quad (-1<x<1)$.
9. $d(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}dx \quad (-1<x<1)$.

10. $d(arctgx) = \frac{1}{1+x^2}dx$.

11. $d(arcctgx) = -\frac{1}{1+x^2}dx$.

12. $d(shx) = chx \cdot dx$.

13. $d(chx) = shx \cdot dx$.

14. $d(thx) = \frac{1}{ch^2 x}dx$.

15. $d(cthx) = -\frac{1}{sh^2 x}dx \quad (x \neq 0)$.

5.51. Differensiallashning sodda qoidalari

$f(x)$ va $g(x)$ funksiyalar (a,b) da aniqlangan bo'lib, $x \in (a,b)$ da $df(x)$, $dg(x)$ mavjud bo'lsin. U holda $f(x) \pm g(x)$, $f(x) \cdot g(x)$ va $\frac{f(x)}{g(x)}$ ($g(x) \neq 0$) ham differensiallanuvchi bo'ladi va ular uchun quyidagi formula o'rinli:

$$d(f(x) \pm g(x)) = df(x) \pm dg(x),\ d(f(x) \cdot g(x)) = g(x)df(x) + f(x)dg(x),$$

$$d\frac{f(x)}{g(x)} = \frac{g(x)df(x) - f(x)dg(x)}{g^2(x)} \quad (g(x) \neq 0).$$

Murakkab funksiyaning differensiali $u = f(x)$ funksiya (a,b) intervalda, $y = F(u)$ esa (c,d) intervalda aniqlangan bo'lib, bu funksiyalar yordamida

$$y = F(f(x)) = \Phi(x)$$

murakkab funksiya tuzilgan bo'lsin.

Murakkab funksiyaning differensiali

$$d\Phi(x) = d(F(f(x))) = (F(f(x)))'dx = F'(u)f'(x)dx = F'(u)du.$$

Misol. Ushbu $y = \sin(x^2 + \cos x)$ funksiyani differensialini toping.

Yechish.

$$dy = (\sin((x^2 + \cos x))'dx = \cos(x^2 + \cos x) \cdot (x^2 + \cos x)'dx =$$
$$= (2x - \sin x) \cdot \cos(x^2 + \cos x)dx.$$

5.52. Funksiya differensiali va taqribiy hisoblash formulalari

Funksiya differensiali taqribiy hisoblash formulalarini topish imkonini beradi.

$f(x)$ funksiya (a,b) intervalda aniqlangan bo'lib, $\forall x_0 \in (a,b)$ nuqtada chekli $f'(x_0) \neq 0$ hosilaga ega bo'lsin va $f(x_0)$ qulay hisoblanadigan bo‘lsa, bu nuqta atrofida funksiya qiymatini quyidagicha taqribiy hisoblash mumkin:

$$f(x) \approx f(x_0) + f'(x_0)x .$$

Bu formuladan quyidagilarga ega bo‘lamiz ($x = 0$ nuqta atrofida):

$$(1+x)^{\mu} \approx 1 + \mu x, \; \sqrt{1+x} \approx 1 + \frac{1}{2}x, \; e^x \approx 1 + x, \; \ln(1+x) \approx x, \; \sin x \approx x, \; tgx \approx x .$$

6§. YUQORI TARTIBLI HOSILA VA DIFFERENSIALLAR

6.1. Funksiyaning yuqori tartibli hosilalari

$y = f(x)$ funksiya x_0 nuqtaning biror atrofida berilgan bo‘lib, shu atrofida $f'(x_0)$ hosilaga ega bo‘lsin. Agar $f'(x_0)$ ham x_0 nuqtada hosilaga ega bo‘lsa, uni $f(x)$ funksiyaning x_0 nuqtadagi ikkinchi tartibli hosilasi deyiladi va

$$y''_{x_0} \text{ yoki } f''(x_0) \text{ yoki } \left.\frac{d^2y}{dx^2}\right|_{x=x_0} .$$

ko‘rinishida yoziladi. Demak,

$$y''_{x_0} = (y')'_{x_0}, \; f''(x_0) = (f'(x))'_{x=x_0}, \; \left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{dy}{dx}\right)_{x=x_0}$$

$f(x)$ funksiyaning uchinchi, to‘rtinchi va h.k. tartibdagi hosilalari xuddi shunga o‘xshash ta’riflanadi.

Umuman, agar $y = f(x)$ funksiyaning $(n-1)$-tartibli $f^{n-1}(x)$ hosilasi x_0 nuqtaning biror atrofida mavjud bo‘lib, $f^{n-1}(x)$ funksiya x_0 nuqtada hosilaga ega bo‘lsa, uni $y = f(x)$ funksiyaning x_0 nuqtadagi n-tartibli hosilasi deyiladi va

$$y^{(n)}_{x=x_0} \text{ yoki } f^{(n)}(x_0) \text{ yoki } \left.\frac{d^ny}{dx^n}\right|_{x=x_0}$$

ko‘rinishlardan biri orqali yoziladi.

Shunday qilib,

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)' \quad \left(\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{n-1}}\right)\right)$$

bo‘ladi.

6.2. Funksiyaning yuqori tartibli differensiali

$y = f(x)$ funksiya x_0 nuqtaning biror atrofida berilgan bo‘lib, shu atrofda ikki marta differensiallanuvchi bo‘lsin. $f(x)$ funksiya differensiali $dy \equiv df(x)$ ning differensiali berilgan funksiyaning ikkinchi tartibli differensiali deyiladi va

$$d^2y \text{ yoki } d^2f(x)$$

ko‘rinishida yoziladi. Demak, $d^2y = d(dy)$ yoki $d^2f(x) = d(df(x))$. Yuqorida keltirilgan funksiyalarning ikkinchi tartibli differensiali quyidagicha izohlanadi;

a) dy faqat x ning funksiyasi deb faraz qilinadi, ya’ni $f'(x)dx$ ning differensiali hisoblanganda dx o‘zgarmas ko‘paytuvchi deb qaraladi.

b) $f'(x)$ ning differensiali hisoblanganda x ning orttirmasi $\Delta x = dx$ ni birinchi tartibli differensiali $dy = f'(x)dx$ ni hisoblangandagi dx ning qiymatiga teng deb qaraladi.

$f(x)$ ning uchinchi, to‘rtinchi va h.k. tartibdagi differensiallari xuddi shunga o‘xshash ta’riflanadi.

Umuman, $f(x)$ funksiyaning n-tartibli differensiali $d^n f(x)$ ni

$$d^n f(x) = d(d^{n-1}f(x)) = f^{(n)}(x)dx^n$$

deb ta’riflanadi.

6.3. Funksiyaning hosilasi bilan uning differensiali orasidagi bog‘lanish

Funksiyaning hosilasi bilan uning differensiali orasida

$$d^n y = y^{(n)}dx^n \text{ yoki } d^n f(x) = f^{(n)}(x)dx^n$$

bog‘lanish mavjud.

6.4. Yuqori tartibli differensiallar uchun sodda qoidalar va asosiy formulalar

$f(x)$ va $g(x)$ funksiyalar x_0 nuqtaning biror atrofida aniqlangan bo‘lib, ular shu atrofda $f^{(n)}(x)$ va $g^{(n)}(x)$ hosilalarga ega bo‘lsin. U holda

1. $d^n(c \cdot f(x)) = c \cdot d^n f(x), \quad c = const$,
2. $d^n(f(x) \pm g(x)) = d^n f(x) \pm d^n g(x)$,
3. $d^n(f(x) \cdot g(x)) = d^n f(x) \cdot g(x) + C_n^1 d^{n-1} f(x) \cdot dg(x) +$
$+ C_n^2 d^{n-2} f(x) \cdot d^2 g(x) + ... + C_n^{n-1} df(x) \cdot d^{n-1} g(x) + f(x) \cdot d^n g(x)$.

(***Leybnis formulasi***) bo‘ladi.

1. $y = a^x$ bo‘lsa, $d^n y = a^x \ln^n a dx^n$.

2. $y = e^x$ bo‘lsa, $d^n y = e^x dx^n$.

3. $y = \sin x$ bo‘lsa, $d^n y = \sin\left(x + n \cdot \frac{\pi}{2}\right) dx^n$.

4. $y = \cos x$ bo‘lsa, $d^n y = \cos\left(x + n \cdot \frac{\pi}{2}\right) dx^n$.

5. $y = \ln x$ bo‘lsa, $d^n y = \frac{(-1)^{n-1}(n-1)!}{x^n} dx^n$.

6. $y = \frac{1}{x}$ bo‘lsa, $d^n y = \frac{(-1)^n n!}{x^{n+1}} dx^n$.

7. $y = x^m$ bo‘lsa, $d^n y = m(m-1)...(m-n+1)x^{m-n} dx^n$.

8. $y = (1+x)^\alpha$ bo‘lsa, $d^n y = \alpha(\alpha-1)...(\alpha-n+1)(1+x)^{\alpha-n} dx^n$.

6.5. Differensial hisobning asosiy teoremalari

1°. *Ferma teoremasi.* $y = f(x)$ funksiya biror X oraliqda aniqlangan va bu oraliqning ichki c nuqtasida o‘zining eng katta (eng kichik) qiymatiga erishsin. Agar shu nuqtada funksiya chekli $f'(c)$ hosilaga ega bo‘lsa, u holda

$$f'(c) = 0$$

bo‘ladi.

2°. *Roll teoremasi.* $y = f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsin. Agar shu funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo‘lib, $f(a) \neq f(b)$ bo‘lsa, u holda shunday c ($a < c < b$) nuqta topiladiki,

$$f'(c) = 0$$

bo‘ladi.

3°. *Lagranj teoremasi.* $y = f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsin. Agar bu funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo‘lib, $f(a) = f(b)$ bo‘lsa, u holda shunday c ($a < c < b$) nuqta topiladiki, bu nuqtada

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \left(f(b) - f(a) = f'(c)(b - a)\right)$$

bo‘ladi.

4°. *Koshi teoremasi.* $f(x)$ va $g(x)$ funksiyalar $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsin. Agar shu funksiyalar (a,b) intervalda chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lib, $\forall x \in (a,b)$ uchun $g'(x) \neq 0$ bo‘lsa, u holda shunday c ($a < c < b$) nuqta topiladiki,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

bo'ladi.

6.6. Teylor formulasi

$y = f(x)$ funksiya (a,b) intervalda aniqlangan bo'lib, $x_0 \in (a,b)$ nuqtada $f'(x), f''(x),..., f^n(x)$ hosilalarga ega bo'lsin. Uni shu nuqta atrofida quyidagi formula bilan ifodalash mumkin:

$$f(x) = f(x_0) + \frac{f(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x). \quad (1)$$

Bu Teylor formulasidir. $R_n(x)$ ga Teylor formulasining qoldiq hadi deyiladi. U quyidagi ko'rinishlarga ega:

1) $R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x - x_0)^{n+1}(1-\theta)^n$, $\quad (c = x_0 + \theta(x - x_0), \quad 0 < \theta < 1)$ (***Koshi ko'rinishi***),

2) $R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x - x_0)^{n+1}$, $(c = x_0 + \theta(x - x_0),\ 0 < \theta < 1)$ (***Lagranj ko'rinishi***),

3) $R_n(x) = \theta((x - x_0)^n)$ (***Peano ko'rinishi***)

6.7. Bir o'zgaruvchili funksiya uchun Makloren formulasi

Agar (1) formulada $x_0 = 0$ deb olsak, unda ushbu

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + o(x^n) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k + o(x^n)$$

formula hosil bo'ladi. Bu ***Makloren formulasi*** deyiladi.

Ushbu $f(x) = e^x$, $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = (1 + x)^m$, $f(x) = \ln(1 + x)$ funksiyalar uchun Makloren formulasi quyidagicha bo'ladi:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + o(x^n)$,

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...+ (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$,

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...+ (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1})$,

4. $(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + ...+ \frac{m(m-1)...(m-n+1)}{n!}x^n + o(x^n)$.

5. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$

7§. DIFFERENSIAL HISOBNING BA’ZI TATBIQLARI

7.1. Funksiyaning o‘suvchiligi va kamayuvchiligi

Funksiya hosilasi yordamida uning o‘suvchiligini hamda kamayuvchiligini aniqlash mumkin.

Teorema. $f(x)$ funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo‘lsin. Bu funksiya shu intervalda o‘suvchi bo‘lishi uchun ixtiyoriy $x \in (a,b)$ uchun

$$f'(x) \geq 0$$

bo‘lishi zarur va yetarli.

Teorema. $f(x)$ funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo‘lsin. Funksiya shu intervalda kamayuvchi bo‘lishi uchun ixtiyoriy $x \in (a,b)$ uchun

$$f'(x) \leq 0$$

bo‘lishi zarur va yetarli.

Misol. Ushbu

$$y = x^4$$

funksiyaning o‘sish va kamayish sohalari topilsin.
Hosilani topamiz: $y' = 4x^3$; $x > 0$ bo‘lganda $y' > 0$ bo‘ladi - funksiya o‘sadi; $x < 0$ bo‘lganda $y' < 0$ bo‘ladi - funksiya kamayadi.

7.2. Funksiyaning o‘zgarmas qiymatni saqlashi

$f(x)$ funksiya (a,b) intervalda aniqlangan bo‘lsin.

Teorema (Ferma teoremasi). $f(x)$ funksiya (a, b) intervalda chekli $f'(x)$ hosilaga ega bo‘lsin. Bu funksiya (a, b) intervalda o‘zgarmas bo‘lishi uchun shu intervalda

$$f'(x) = 0$$

bo‘lishi zarur va yetarli.

1-natija. Agar $f(x)$ va $g(x)$ funksiyalar (a,b) intervalda chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lib, shu intervalda

$$f'(x) \equiv g'(x)$$

tenglik o‘rinli bo‘lsa, u holda $f(x)$ bilan $g(x)$ funksiyalar (a,b) intervalda bir-biridan o‘zgarmas songa farq qiladi:

$$f(x) \equiv g(x) + C, \ C = const.$$

Eslatma. $f(x)$ funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo‘lib, bu funksiyaning (a,b) da ***qat’iy o‘suvchi* (*qat’iy kamayuvchi*)**

bo‘lishidan, $f'(x)$ ning ixtiyoriy $x \in (a, b)$ da musbat (manfiy) bo‘lishi har doim kelib chiqavermaydi.

Masalan, $f(x) = x^3$ funksiyani qaraylik. Bu funksiyaning R da qat’iy o‘suvchi. Bu funksiya $f'(x) = 3x^3$ bo‘lib, $x = 0$ nuqtada $f'(0) = 0$.

7.3. Funksiyaning ekstremum qiymatlari

$f(x)$ funksiya (a, b) intervalda aniqlangan bo‘lib, $x_0 \in (a, b)$ bo‘lsin.

Ta’rif. Agar $x_0 \in (a, b)$ nuqtaning shunday atrofi $U_\delta(x_0)$ mavjud bo‘lsaki, $\forall x \in U_\delta(x_0)$ uchun

$$f(x) \le f(x_0) \quad (f(x \ge f(x_0))$$

tengsizlik o‘rinli bo‘lsa, u holda $f(x)$ funksiya x_0 nuqtada ***maksimumga (minumumga)*** erishadi deyiladi. $f(x_0)$ qiymat $f(x)$ funksiyaning $U_\delta(x_0)$ dagi ***maksimumi (minimumi)*** deyiladi.

2- ta’rif . Agar $x_0 \in (a, b)$ nuqtaning shunday atrofi $U_\delta(x_0) \subset (a, b)$ bo‘lsaki, ixtiyoriy $x \in U'_\delta(x_0)$ uchun $f(x) < f(x_0)$ $(f(x) > f(x_0))$ tengsizlik o‘rinli bo‘lsa, u holda x_0 nuqtada $f(x)$ funksiya ***qat’iy maksimumga*** (***qat’iy minumumga***) erishadi deyiladi, $f(x_0)$ qiymat $f(x)$ funksiyaning $U_\delta(x_0)$ dagi ***qat’iy maksimumi (minimumi)*** deyiladi.

Funksiyaning $U_\delta(x_0)$ dagi maksimum (minimum) qiymatlari

$$f(x_0) = \max_{x \in U_\delta(x_0)} \{f(x)\} \qquad f(x_0) = \min_{x \in U_\delta(x_0)} \{f(x)\}$$

kabi belgilanadi. Bunda max (min) lotincha maximum (minimum) so‘zidan olingan bo‘lib, eng katta (eng kichik) degan ma’noni anglatadi.

Funksiyaing maksimumi va minimumi umumiy nom bilan uning ***ekstremumi*** deb ataladi.

Eslatma. Yuqoridagi ta’riflarda $f(x)$ funksiyaning $x_0 \in (a,b)$ dagi qiymati uning shu nuqta $f(x_0)$ atrofidan olingan nuqtalardagi qiymatlari bilangina taqqoslandi. Shuning uchun funksiyaning ekstremumi lokal ekstremum deb yuritiladi.

Eslatma. $f(x)$ funksiya (a,b) intervalda bir nechta maksimum va minumumlarga ega bo‘lishi mumkin.

Funksiya hosilalari yordamida uning ekstremumlari hamda funksiyaga ekstremum qiymat beradigan nuqtalar topiladi.

7.4. Ekstremumning zaruriy sharti

$f(x)$ funksiya (a,b) intervalda aniqlangan bo‘lib, $x_0 \in (a,b)$ nuqtada maksimum (minimum) ga erishsin. x_0 nuqtaning shunday $U_\delta(x_0) \subset (a,b)$ atrofi topiladiki, ixtiyoriy $x \in U_\delta(x_0)$ da $f(x_0) \geq f(x)$ $(f(x_0) \leq f(x))$ tengsizlik o‘rinli bo‘ladi.

Funksiyaning x_0 nuqtadagi hosilasi haqida, umuman aytganda, quyidagi uch hol bo‘lishi mumkin:

1) $f'(x_0)$ mavjud va chekli,
2) $f'(x_0)$ mavjud va cheksiz.
3) Hosila mavjud emas.

Birinchi holda Ferma teoremasiga ko‘ra $f'(x_0) = 0$ bo‘ladi (7.2 bandga qarang).

$f(x)$ funksiya uchun biror $x^* \in (a,b)$ nuqtada chekli hosila mavjud va $f'(x^*) = 0$ bo‘lishidan uning x^* nuqtada ekstremumga ega bo‘lishi har doim kelib chiqavermaydi. Masalan, $f(x) = x^3$ funksiya uchun $f'(x) = 3x^2$ va $x = 0$ nuqtada $f'(x_0) = 0$ bo‘lsa ham u $x = 0$ nuqtada ekstremumga ega emas (bu funksiya qat’iy o‘suvchi ekanligi bizga ma’lum).

Demak, Ferma teoremasi funksiya ekstremumga erishishining zaruriy shartini ifodalaydi.

Ikkinchi holning esa bo‘lishi mumkin emas. Agarda $f'(x_0) = +\infty$ $(-\infty)$ bo‘lsa, $f(x)$ funksiya x_0 nuqtaning atrofida o‘suvchi (kamayuvchi) bo‘ladi. Haqiqatan hosila ta’rifidan $\delta > 0$ topiladiki, $0 < x - x_0 < \delta$ bo‘lgan x lar uchun $\frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{\varepsilon}$, ya’ni $f(x) > f(x_0)$ tengsizlik kelib chiqadi. Xuddi shuningdek, $-\delta < x - x_0 < 0$ bo‘lgan x lar uchun $f(x) < f(x_0)$ tengsizlik kelib chiqadi.

Uchinchi hol, biz $f(x) = |x|$ funksiyaning $x = 0$ nuqtada hosilasi mavjud emasligini ko‘rgan edik. Bu funksiya $x = 0$ nuqtada minimumga ega bo‘lishi ravshandir. Demak, funksiya hosilaga ega bo‘lmagan nuqtalarda ham ekstrimumga erishishi mumkin.

7.5. Funksiyaning statsionar nuqtaslari

Funksiya hosilasini nolga aylantiradigan nuqtalar funksiyaning ***statsionar*** (turg‘un, kritik) nuqtalari deb ham ataladi.

7.6. Ekstremumning yetarli shartlari

$f(x)$ funksiya x_0 nuqtada uzluksiz bo‘lib, uning

$$\dot{U}_{\delta}(x_0) = \{x; x \in R, x_0 - \delta < x < x_0 + \delta \ x \neq x_0\}$$

atrofida chekli $f'(x)$ hosilaga ega bo‘lsin.

a) Agar

$$\forall x \in \dot{U}^{-}_{\delta}(x_0) \ \ uchun \ f'(x) > 0,$$

$$\forall x \in \dot{U}^{+}_{\delta}(x_0) \ \ uchun \ f'(x) < 0$$

tengsizliklar o‘rinli bo‘lsa, ya’ni $f'(x)$ hosila x_0 nuqtadan o‘tishida o‘z ishorasini “+” dan “-” ga o‘zgartirsa, u holda $f(x)$ funksiya x_0 nuqtada ***maksimumga*** erishadi.

b) Agar

$$\forall x \in \dot{U}^{-}_{\delta}(x_0) \text{ uchun } (f(x))' < 0$$

$$\forall x \in \dot{U}^{+}_{\delta}(x_0) \text{ uchun } (f(x))' > 0$$

tengsizliklar o‘rinli bo‘lsa, ya’ni $f'(x)$ hosila x_0 nuqtani o‘tishda o‘z ishorasini “—” dan “+” ga o‘zgartirsa, u holda $f(x)$ funksiya x_0 nuqtada ***minimumga*** erishadi.

v) Agar

$$\forall x \in \dot{U}^{-}_{\delta}(x_0) \ \ uchun \ f'(x) > 0,$$

$$\forall x \in \dot{U}^{+}_{\delta}(x_0) \ \ uchun \ f'(x) > 0$$

yoki

$$\forall x \in \dot{U}^{-}_{\delta}(x_0) \quad uchun \quad f'(x) < 0,$$

$$\forall x \in \dot{U}^{+}_{\delta}(x_0) \quad uchun \quad f'(x) < 0$$

tengsizliklar o‘rinli bo‘lsa, ya’ni $f'(x)$ hosila x_0 nuqtani o‘tishda o‘z ishorasini o‘zgartirmasa, u holda $f(x)$ funksiya x_0 nuqtada ***ekstremumga*** erishmaydi, u holda $f(x)$ funksiya x_0 nuqtaning $U'_{\delta}(x)$ atrofida ***qat’iy o‘suvchi (yoki qat’iy kamayuvchi)*** bo‘ladi.

7.7. Funksiya ekstremumini topishda uning yuqori tartibli hosilalaridan foydalanish

$f(x)$ funksiya $x_0 \in (a,\ b)$ nuqtada $f', f'',..., f^{(n)}$ hosilalarga ega bo‘lib, biror $n \geq 2$ son uchun

$$f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0, \ \ f^{(n)}(x_0) \neq 0$$

bo‘lsin.

a) Agar n - juft son, ya’ni $n = 2m(m \in N)$ bo‘lib,

$$f^{(n)}(x_0) = f^{(2m)}(x_0) < 0$$

tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya x_0 nuqtada ***maksimumga***,

$$f^{(n)}(x_0) = f^{(2m)}(x_0) > 0$$

tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya x_0 nuqtada ***minimumga*** erishadi.

b) Agar n -toq son, ya’ni $n = 2m + 1$ $(m \in N)$ bo‘lsa, $f(x)$ funksiya x_0 nuqtada ***ekstremumga*** erishmaydi.

Natija. Agar $x = 0$ nuqtada $f(x)$ funksiyaning statsionar nuqtasi bo‘lib, $f(x)$ funksiya x_0 nuqtada chekli $f''(x_0) \neq 0$ hosilaga ega bo‘lsa, $f''(x_0) < 0$ bo‘lganda $f(x)$ funksiya x_0 nuqtada maksimumga, $f''(x_0) > 0$ bo‘lganda $f(x)$ funksiya x_0 nuqtada minimumga ega bo‘ladi.

7.8. Funksiyaning eng katta va eng kichik qiymatlari

$f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsin. Veyershtrassning ikkinchi teoremasiga ko‘ra, funksiyaning $[a,b]$ da eng katta hamda eng kichik qiymatlari mavjud bo‘ladi va bu qiymatlarga $[a,b]$ segmentning nuqtalarida erishadi. Funksiyaning eng katta qiymati quyidagicha topiladi:

1) $f(x)$ funksiyaning (a,b) intervaldagi maksimum qiymatlari topiladi. Funksiyaning hamma maksimum qiymatlaridan iborat to‘plam $\{\max f(x)\}$ bo‘lsin.

2) Funksiyaning $[a,b]$ segmentning chegaralaridagi, ya’ni $x = a,\ x = b$ nuqtalardagi $f(a)$ va $f(b)$ qiymatlari hisoblanadi. So‘ngra $\{\max f(x)\}$ to‘plamning barcha elementlari bilan $f(a)$ va $f(b)$ lar taqqoslanadi. Bu qiymatlar ichida eng kattasi $f(x)$ funksiyaning $[a,b]$ segmentdagi eng katta qiymati bo‘ladi.

3) $f(x)$ funksiyaning (a,b) intervalidagi barcha minimum qiymatlari topilib, ulardan $\{\min f(x)\}$ to‘plam tuziladi.

4) $[a,b]$ segmentning chegaralari $x = a,\ x = b$ nuqtalarda $f(x)$ funksiyaning $f(a)$, $f(b)$ qiymatlari ichida eng kichigi $f(x)$ funksiyaning $[a,b]$ segmentdagi eng kichik qiymati bo‘ladi.

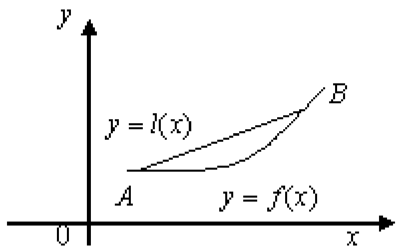
7.9. Funksiyaning botiqligi

$f(x)$ funksiya (a,b) intervalda aniqlangan bo‘lib, bu intervaldan olingan $x_1 \in (a,b),\ x_2 \in (a,b)$ nuqtalar uchun $l(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$ ni qaraymiz, bu yerda $x_1 < x_2$. Ravshanki, $(x_1, x_2) \subset (a,b)$.

Ta’rif. Agar har qanday $(x_1, x_2) \subset (a,b)$ olinganda ham $\forall x \in (x_1, x_2)$ uchun

$$f(x) \le l(x) \quad (f(x) < l(x))$$

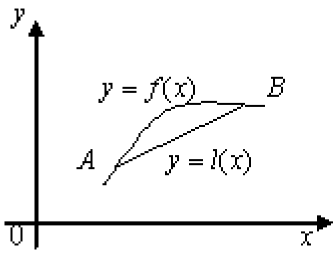
tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya (a,b) intervalda botiq (qat’iy botiq) funksiya deyiladi.



$f(x)$ funksiya (a,b) intervalda aniqlangan bo‘lib, bu intervalda chekli $f'(x)$ hosilaga ega bo‘lsin.

Teorema. $f(x)$ funksiya (a,b) intervalda botiq (qat’iy botiq) bo‘lishi uchun uning $f'(x)$ hosilasining (a,b) da o‘suvchi (qat’iy o‘suvchi) bo‘lishi zarur va yetarli.

7.10. Funksiyaning qavariqligi



$f(x)$ funksiya (a,b) intervalda aniqlangan bo‘lib, shu intervaldan olingan $x_1 \in (a,b),\ x_2 \in (a,b)$ nuqtalar uchun $l(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$ ni qaraymiz, bu yerda $x_1 < x_2$. Ravshanki, $(x_1, x_2) \subset (a,b)$.

Ta’rif. Agar har qanday $(x_1, x_2) \subset (a,b)$ olinganda ham ixtiyoriy $x \in (x_1, x_2)$ uchun

$$f(x) \ge l(x) \quad (f(x) > l(x))$$

tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya (a,b) intervalda ***qavariq (qat’iy qavariq) funksiya*** deyiladi.

Teorema. $f(x)$ funksiyaning (a,b) intervalda qavariq (qat’iy qavariq) bo‘lishi uchun uning $f'(x)$ hosilasining (a,b) da kamayuvchi bo‘lishi zarur va yetarli.

Funksiyaning qavariq hamda botiqligini uning ikkinchi tartibli hosilasidan ham (agar u mavjud bo‘lsa) foydalanib tekshirish mumkin.

Teorema. $f(x)$ funksiyaning (a,b) intervalda botiq (qavariq) bo‘lishi uchun shu intervalda

$$f''(x)\geq 0 \quad (f''(x)\leq 0)$$

tengsizlik o‘rinli bo‘lishi zarur va yetarli.

7.11. Funksiyaning egilish nuqtalari

$f(x)$ funksiya x_0 nuqtaning $U_\delta(x_0)$ atrofida aniqlangan bo‘lsin.

Ta’rif. Agar $f(x)$ funksiya $U^-{}_\delta(x_0)$ oraliqda botiq (qavariq) bo‘lib, $U^+{}_\delta(x_0)$ oraliqda esa qavariq (botiq) bo‘lsa, u holda x_0 nuqta funksiyaning (funksiya grafigining) ***egilish nuqtasi*** deb ataladi.

7.12. Funksiya grafigining asimptotalari

$f(x)$ funksiya $a\in R$ nuqtaning biror atrofida aniqlangan bo‘lsin.

Ta’rif. Agar ushbu

$$\lim_{x\to a+0} f(x),\ \lim_{x\to a-0} f(x)$$

limitlardan biri yoki ikkalasi cheksiz bo‘lsa, u holda $x=a$ to‘g‘ri chiziq $f(x)$ funksiya grafigining ***vertikal asimptotasi*** deb ataladi.

Masalan, $y=\frac{1}{x}$ funksiya grafigi uchun $x=0$ to‘g‘ri chiziq vertikal asimptota bo‘ladi.

Endi $y=f(x)$ funksiya (a,∞) $((-\infty,a))$ oraliqda aniqlangan bo‘lsin.

Ta’rif. Agar shunday o‘zgarmas k va b sonlar mavjud bo‘lsaki, $x\to+\infty$ da $f(x)$ funksiya ushbu

$$f(x)=kx+b+\alpha(x)$$

ko‘rinishda ifodalansa (bunda $\lim\limits_{x\to+\infty}\alpha(x)=0$), u holda $y=kx+b$ to‘g‘ri chiziq $f(x)$ funksiya grafigining ***og‘ma asimptotasi*** deb ataladi.

Masalan,

$$f(x)=\frac{2x^2-4x+5}{x+2}$$

bo‘lsin. Bu funksiyani

$$f(x)=2x-8+\frac{21}{x+2}$$

ko‘rinishda yozish mumkin. Demak, $x\to+\infty$ da $\alpha(x)=\frac{2}{x+1}\to 0$ bo‘lib, berilgan funksiya $f(x)=2x-8+\alpha(x)$ ko‘rinishda ifodalanadi. Bundan esa

$f(x) = 2x - 8$ to‘g‘ri chiziq funksiya grafigining og‘ma asimptotasi ekanligi kelib chiqadi.

Teorema. $f(x)$ funksiya grafigi $y = kx + b$ og‘ma asimptotaga ega bo‘lishi uchun

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k, \quad \lim_{x \to \infty} [f(x) - kx] = b$$

tengliklarning o‘rinli bo‘lishi zarur va yetarli.

7.13. Funksiyani tekshirish. Grafiklarni yasash

Funksiyalarni tekshirish va ularning grafiklarini yasashni quyidagi sxema bo‘yicha olib borish maqsadga muvofiqdir:

1^0. Funksiyaning aniqlanish sohasini topish;

2^0. Funksiyani uzluksizlikka tekshirish va uzilish nuqtalarini topish;

3^0. Funksiyani juft, toq hamda davriyligini aniqlash;

4^0. Funksiyani monotonlikka tekshirish;

5^0. Funksiyani ekstrimumga tekshirish;

6^0. Funksiya grafigining qavariq hamda botiqligini aniqlash, egilish nuqtalarini topish;

7^0. Funksiya grafiginng asimptotalarini topish;

8^0. Funksiyaning haqiqiy nollarini (agar ular mavjud bo‘lsa), shuningdek argument x ning bir nechta xarakterli qiymatlarida qiymatlarini topish.

Misol. $y = x + e^{-x}$ funksiyani to‘liq tekshiring va grafigini chizing.
Funksiyani yuqorida ko‘rsatilgan sxema asosida to‘liq tekshiramiz.
Funksiyaning aniqlanish sohasi haqiqiy sonlar to‘plami.
Funksiya juft ham, toq ham, davriy ham emas.
Funksiya uzluksiz. OY o‘qi bilan kesishish nuqtasi: $y = f(0) = 1$.
OX o‘qi bilan kesishmaydi. Endi funksiyani monotonlik va ekstremumga tekshiramiz:

$$y' = \left(x + e^{-x}\right)' = 1 - e^{-x} = 0 \Rightarrow x = 0 \,.$$

Intervallar usulidan foydalanib bu ifodaning ishorasi saqlanadigan oraliqlarni topamiz va quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;+\infty)$
y'	-	0	+
y	↘	$\min f(0) = 1$	↗

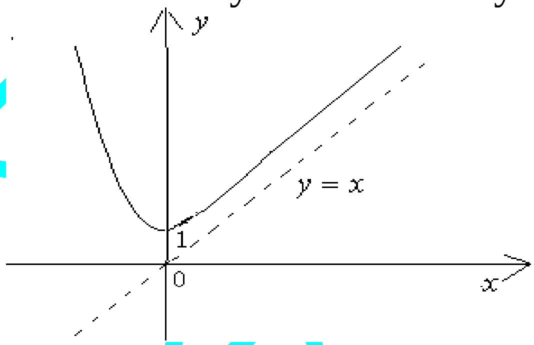
Qavariqlikka tekshirish uchun y'' ni hisoblaymiz:

$$y''=(y')'=\left(1-e^{-x}\right)'=e^{-x}\neq 0\Rightarrow\frac{1}{e^{x}}>0\ .$$

Funksiya hamma joyda botiq.
Funksiya asimptotalarini topamiz:
a) Vertikal asimptota: yo‘q
б) Gorizontal asimptota: $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\left(x+e^{-x}\right)=\infty\Rightarrow$ Gorizontal asimptota yo‘q.
в) Og‘ma asimptota: $k=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\infty}\dfrac{x+e^{-x}}{x}=1$,

$$b=\lim_{x\to\infty}[f(x)-kx]=\lim_{x\to\infty}\left(x+e^{-x}-x\right)=0\Rightarrow y=x$$

og‘ma asimptota.
Endi topilgan ma'lumotlardan foydalanib funksiya grafigini chizamiz



7.14. Aniqmasliklarni ochish. Lopital qoidalari

Funksiyalarning limitini o‘rganish jarayonida $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ ko‘rinishdagi aniqmasliklarni ochish bilan shug‘ullanishga to‘g‘ri keladi. Hosilalardan foydalanib aniqmasliklarni ochish ***Lopital qoidalari*** deyiladi.

1^0. $\frac{0}{0}$ ko‘rinishidagi aniqmaslik.

Ma’lumki, $x\to a$ da $f(x)\to 0$, $g(x)\to 0$ bo‘lsa, $\dfrac{f(x)}{g(x)}$ nisbat $\dfrac{0}{0}$ ko‘rinishidagi aniqmaslikni ifodalaydi. Ko‘pincha $x\to a$ da $\dfrac{f(x)}{g(x)}$ nisbatning limitini topishga qaraganda $\dfrac{f'(x)}{g'(x)}$ nisbatning limitini topish oson bo‘ladi. Bu nisbatlar limitlarining tengligini quyidagi teorema ko‘rsatadi.

Teorema. (a,b) intervalda aniqlangan, uzluksiz $f(x)$ va $g(x)$ funksiyalar uchun ushbu shartlar bajarilgan bo‘lsin:

1) $\lim\limits_{x\to a} f(x)=0,\ \lim\limits_{x\to a} g(x)=0;$

2) (a,b) da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

3) $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=k$ (k-chekli yoki cheksiz).

U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$$

tenglik o‘rinli bo‘ladi.

Lopital qoidasidan foydalanib, $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ limitni osonlik bilan isbotlash mumkin.

Haqiqatan,

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$$

2^0. $\dfrac{\infty}{\infty}$ ko‘rinishidagi aniqmaslik.

Ma’lumki, $x\to a$ da $f(x)\to\infty$ $g(x)\to\infty$ bo‘lsa, $\dfrac{f(x)}{g(x)}$ nisbat $\dfrac{\infty}{\infty}$ ko‘rinishidagi aniqmaslikni ifodalaydi. Bunday aniqmaslikni ochishda $f(x)$ va $g(x)$ funksiyalarning hosilalaridan foydalanish mumkin.

Teorema. (a,b) intervalda $f(x)$ va $g(x)$ funksiyalar uchun ushbu shartlar bajarilsin:

1) $\lim\limits_{x\to a} f(x)=\infty,\ \lim\limits_{x\to a} g(x)=\infty$;

2) (a,b) intervalda chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

3) $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=k$ (k-chekli yoki cheksiz). U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$$

tenglik o‘rinli bo‘ladi.

3^0. $0\cdot\infty$ ko‘rinishidagi aniqmaslik.

$\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=\infty$ bo‘lganda $f(x)\cdot g(x)$ ifoda $0\cdot\infty$ ko‘rinishdagi aniqmaslik bo‘lib, uni quyidagi

$$f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

kabi yozish orqali $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishdagi aniqmaslikka keltirish mumkin.

4^0. $\infty-\infty$ ko‘rinishidagi aniqmaslik.

$\lim\limits_{x\to a} f(x)=+\infty$, $\lim\limits_{x\to a} g(x)=+\infty$ bo‘lganda $f(x)-g(x)$ ifoda $\infty-\infty$ ko‘rinishdagi aniqmasliklarni ochishda, ularni $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishdagi aniqmaslikka keltirib hisoblanadi.

5^0. $1^\infty, 0^\infty, \infty^0$ ko‘rinishidagi aniqmasliklar.

Aytaylik,

$$\lim_{x\to a}[g(x)\cdot \ln f(x)]=b$$

(b-chekli yoki cheksiz) bo‘lsin deylik. Unda

$$\lim_{x\to a}[f(x)]^{g(x)}=\lim_{x\to a} e^{g(x)\ln f(x)}=e^b$$

bo‘ladi.

Eslatma. Agar $f(x)$ va $g(x)$ funksiyalarning $f'(x)$ va $g'(x)$ hosilalari ham $f(x)$ va $g(x)$ lar singari yuqorida keltirilgan teoremalarning barcha shartlarini qanoatlantirsa, u holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

tengliklar o‘rinli bo‘ladi, ya’ni bu holda Lopital qoidasini takror qo‘llash mumkin bo‘ladi.

M4. Hosila va differensiallashga doir mashqlar

1.Hosilalar jadvali va qoidalari yordamida quyidagi funksiyalar hosilalarini toping

1) $y=x^5-\frac{1}{4}x^4+\frac{1}{3}x^3+3{,}5x+0{,}15$

2) $y=3\sqrt{x}-\frac{3}{x}+5^{10}$

3) $y=x^{2025}+\sin 25+\frac{5}{x^2}+\frac{1}{7}$

4) $y=x^{\sqrt{3}}-\sqrt{2}\,x^{\sqrt{2}}-\sqrt{7}$

5) $y=2x\sin x$

6) $y=x^3\cos x$

7) $y=(x^2+x)\operatorname{tg} x$ 8) $y=e^x3^x+5^x$

9) $y=\frac{5x+6}{7x-4}$ 10) $y=5^x\log_6 x$

11) $y=\frac{\sqrt{x}-1}{\sqrt{x}+1}$ 12) $y=\frac{x^3}{\sin x}$

13) $y=x^3\cos x+x\sin 5$ 14) $y=6x^4\cdot 2^x$

15) $y=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}+\frac{5}{\sqrt[4]{x}}$ 16) $y=\sqrt[3]{x}\sin x$

2. Murakkab funksiya hosilalarini toping

1) $y=(5x^2+2x-1)^5$ 2) $y=\sin\frac{5}{6}x$

3) $y=2^{x^2-x}$ 4) $y=\operatorname{tg}6x^2$

5) $y=\sin(\sin x)$ 6) $y=\cos(\cos x)$

7) $y=e^{\cos x}+e^{\sin x}-5$ 8) $y=\ln(x^2+5x-1)$

9) $y=\arcsin\ sinx$ 10) $y=\log_2(\sqrt{x+\sqrt{x}})$

11) $y=sin^5(x^2+x)$ 12) $y=log_2^6\sin x$

13) $y=\ln(1-\sqrt{1-x})$ 14) $y=\sqrt[6]{x^2+\sin x}$

15) $y=2^{\sin x}+e^{-x}$ 16) $y=(\cos x)^{\sin x}$

3. Parametrga bog'liq funksiyaning birinchi va ikkinchi tartibli hosilalari topilsin

1) $x=\sin t\ ,\ y=\cos t$ 2) $x=\cos t\ ,\ y=\sin t$

3) $x=t\sin t\ ,\ y=t\cos t$ 4) $x=1-t^2\ ,\ y=1+t^3$

5) $x=\sqrt{t}\ ,\ y=\sqrt[3]{t}$

4. Quyidagi funksiyalarning ikkinchi tartibli hosilasi topilsin

1) $y=e^{x^2}$ 2) $y=e^{\sin x}$

3) $y=e^{-\sqrt{x}}$ 4)$y=x^3e^{2x}$

5) $y=sin^2x$ 6) $y=x(\cos\ln x+\sin\ln x)$

7) $y=(1+x^2)arc\operatorname{ctg}x$

5. Quyidagi funksiyalarning differensiali topilsin

1) $y=\sin x^2$ 2) $y=\ln\ln x$

3) $y=5^{2x}+2^{x^2}$ 4) $y=e^{\sqrt{\frac{1+x}{1-x}}}$

5) $y=tg^5sinx$ 6) $y=\frac{e^{2x}-1}{e^{2x}+1}$

7) $y=x^x+x^{x^2}$ 8) $y=arc\operatorname{tg}\sqrt{x}$

6. Differensialdan foydalanib quyidagi funksiyalarning berilgan nuqtadagi taqribiy qiymatini toping

1) $y=\sin x\ ,\ x=31°$ 2) $y=\sqrt[3]{x}\ ,\ x=28$

3) $y=\operatorname{ctg}x\ ,\ x=46°30'$ 4) $y=\sqrt{x}\ ,\ x=122$

5)$y=\lg x\ ,\ x=9$

7. Quyidagi funksiyalarning ikkinchi tartibli differensialini toping

1) $y=\cos x^2$ 2) $y=\ln\sin x$

3) $y=e^{2x}+2^{3x}$ 4) $y=sin^2\sqrt{x}$

8. Quyidagi funksiyalarni n-tartibli differensialini toping

1) $y=a^{2x}$ 2) $y=e^{2x}$ 3) $y=\sin 3x$ 4) $y=\cos 3x$

5)$y=\ln 5x$ 6)$y=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

9. Quyidagi funksiyalarning Teylor formulasi bo'yicha x_0 nuqtaning atrofida n-hadgacha yozing(n=4).

1) $f(x) = \sqrt{x}$, $x_0 = 16$ 2) $f(x) = x^2 \ln x$, $x_0 = 1$

3) $f(x) = \operatorname{tg} x$, $x_0 = 0$ 4) $f(x) = \frac{e^x + e^{-x}}{2}$, $x_0 = 0$

5) $f(x) = \ln(3x+1)$, $x_0 = \frac{1}{3}$

10. Quyidagi funksiyalarni Makloren formulasi bo'yicha $O(x^4)$ ha(dgacha yoying.

1) $f(x) = cos^2x$ 2) $f(x) = arcsinx$

3) $f(x) = 2^{3-x}$ 4) $f(x) = \frac{x^3}{x-1}$

11. Quyidagi funksiyalarni monotonlik oraliqlarini aniqlang.

1) $y = 2x - x^3$ 2) $f(x) = \frac{3x}{1+3x}$

3) $f(x) = x + cosx$ 4) $f(x) = -x + sinx$

5) $f(x) = 5x^4 - x^5$ 6) $f(x) = (x-2)^2(x+3)^2$

7) $f(x) = xe^{-5x}$ 8) $f(x) = x^3e^{-x^3}$

9) $f(x) = x^{2lnx}$ 10) $f(x) = (1+\frac{1}{x})^x$

12. Quyidagi funksiyalarning ekstremumlari topilsin.

1) $y = 25x - x^2$ 2) $y = x^2 + 2x - 3$

3) $y = \frac{x^3}{3} + x^2$ 4) $y = x^3 + 2x^2 - 3x$

5) $y = \frac{5x^2}{x-2}$ 6) $y = xe^{-\frac{x^2}{2}}$

7) $y = 2x + ctgx$, $(0, \pi)$ oraliqda

8) $y = x + \frac{1}{x}$ 9) $y = (1-x^2)(1-x^3)$

10) $y = x + 2\sqrt{-x}$ 11) $y = \sqrt{1 - sinx}$

13. Quyidagi funksiyalarning $[a,\ b]$ oraliqda eng katta va eng kichik qiymatlarini toping (agar kesma berilgan bo'lmasa aniqlanish sohasidagi eng katta va eng kichik qiymatini toppish kerak).

1) $y = \frac{5x}{1+x^2}$ 2) $y = x^3$, $[-1; 3]$

3) $y = \sqrt{x(5-x)}$ 4) $y = 2x^3 + 3x^2 - 12x + 1$

5) $y = sin^4x + cos^4x$, $[-1;\ 5]$

6) $y = arccosx$, $[-10;\ 12]$

14. Quyidagi funksiyalarning qavariqlik va botiqlik oraliqlarini toping:

1) $f(x) = x^5 \quad (x > 0)$ 2) $f(x) = e^x$

3) $f(x) = x^5 - 10x^2 + 3x$ 4) $f(x) = \frac{5}{1-x^2}$

5) $f(x) = \frac{3\sqrt{x}}{1+x}$ 6) $f(x) = x + sinx$

7) $f(x) = x + cosx$

15. Quyidagi funksiyalarning egilish nuqtalarini toping.

1) $f(x) = sinx$ 2) $f(x) = e^{\frac{6}{x}}$

3) $f(x) = e^{2x-x^3}$ 4) $f(x) = 4x^2 + \frac{1}{x}$

5) $f(x) = e^{sinx}$ 6) $f(x) = \sqrt{4 - x^2}$

16. Quyidagi funksiyalarni to'liq tekshiring va grafigini chizing.

1) $f(x) = \frac{x}{1+x^2}$ 2) $f(x) = \frac{3x}{x^2-4}$

3) $f(x) = x^2(x^2 - 1)^3$ 4) $f(x) = \frac{5}{x} + 4x^2$

5) $f(x) = \frac{7}{e^x-1}$ 6) $f(x) = x + \sin x$

7) $f(x) = x^2\sqrt{x + 1}$ 8) $f(x) = x(x + 1)^{\frac{3}{2}}$

9) $f(x) = x^2 e^x$ 9) $f(x) = x^2 - 2\ln x$

17. Lopital qoidasidan foydalanib quyidagi limitlar topilsin.

1) $\lim\limits_{x\to 0}\frac{e^x}{\sin 2x}$ 2) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$

3) $\lim\limits_{x\to 0}\frac{x-sinx}{x^3}$ 4) $\lim\limits_{x\to +\infty}\frac{e^x}{x^3}$

5) $\lim\limits_{x\to 0}\frac{\operatorname{tg} x-sinx}{x-sinx}$ 6) $\lim\limits_{x\to 0}\frac{\ln x}{\operatorname{ctg} x}$

7) $\lim\limits_{x\to 0} x\ln x$ 8) $\lim\limits_{x\to +\infty} x^n e^{-x}$

9) $\lim\limits_{x\to 0} x^x$ 10) $\lim\limits_{x\to \infty}(1 + \frac{5}{x})^x$

11) $\lim\limits_{x\to 0}(sinx)^{tgx}$ 12) $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$

18. *shx* funksiyaning Makloren formulasidagi 4-hadi bo'lmagan ifodani aniqlang

1) $\frac{x^6}{6!}$ 2) $\frac{x^4}{4!}$ 3) $\frac{x^3}{3!}$

A) 1,2 B) 1,3 C) 2,3 D)1 E) to'g'ri javob yo'q

19. $\frac{1}{1+x^2}$ funksiya Makloren formulasidagi 3-hadi bo'lmagan ifodani aniqlang

1) $-x^6$ 2) $-x^3$ 3) $-x^2$

A) 1,2 B) 1,3 C) 2,3 D) 1 E) to'g'ri javob yo'q

20. Noto'g'ri javobni aniqlang.

1) $(\sin^2 x)' = \sin 2x$ 2) $(\cos^2 x)' = \cos 2x$ 3) $(\tan x)' = \frac{1}{\sin^2 x}$

A) 1,2 B) 1,3 C) 2,3 D) 1 E) to'g'ri javob yo'q

8-§. ANIQMAS INTEGRAL

8.1. Boshlang‘ich funksiya

$f(x)$ funksiya (a,b) (chekli yoki cheksiz) intervalda aniqlangan bo‘lsin.

Ta’rif. Agar ixtiyoriy $x \in (a,b)$ da $f(x)$ funksiya shu intervalda differensiallanuvchi $F(x)$ funksiyaning hosilasiga teng, ya’ni

$$F'(x) = f(x)$$

bo‘lsa, u holda $F(x)$ funksiya (a,b) intervalda $f(x)$ funksiyaning ***boshlang‘ich funksiyasi*** deyiladi.

8.2. Funksiya differensiali orqali boshlang‘ich funksiya ta’rifi

$f(x)$ funksiya (a,b) (chekli yoki cheksiz) intervalda aniqlangan bo‘lsin.

Ta’rif. Agar ixtiyoriy $x \in (a,b)$ da $f(x)dx$ ifoda shu intervalda differensiallanuvchi $F(x)$ funksiyaning differensialiga teng, ya’ni

$$dF(x) = f(x)dx$$

bo‘lsa, u holda $F(x)$ funksiya (a,b) intervalda $f(x)$ funksiyaning ***boshlang‘ich funksiyasi*** deyiladi.

Misol. $f(x) = x^4$ funksiyaning $(-\infty;+\infty)$ intervalda boshlang‘ich funksiyasi $F(x) = \frac{x^5}{5} + C$ bo‘ladi, bu yerda C o‘zgarmas son.

8.3. Aniqmas integral

Teorema. Agar $f(x)$ funksiya $R = (-\infty;+\infty)$ oraliqda uzluksiz bo‘lsa, $f(x)$ shu oraliqda har doim boshlang‘ich funksiyaga ega bo‘ladi.

$f(x)$ funksiya (a,b) (chekli yoki cheksiz) intervalda aniqlangan bo‘lsin.

Ta’rif. (a,b) intervalda berilgan $f(x)$ funksiya boshlang‘ich funksiyalarining umumiy ifodasi $F(x)+C$ $(C = const)$, shu $f(x)$ funksiyaning aniqmas integrali deyiladi va

$$\int f(x)dx$$

ko‘rinishda belgilanadi. Bunda $\int$ – integral belgisi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ esa integral ostidagi ifoda deyiladi. Demak,

$$\int f(x)dx = F(x) + C \quad (C = const).$$

Misol. Masalan

$$\int x^5 dx = \frac{x^6}{6} + C$$

bo‘ladi.

$\int x^5 dx$ integral shunday funksiyaki,

$$F(x) = \frac{x^6}{6} + C$$

bo‘lib

$$F'(x) = \left(\frac{x^6}{6} + C\right)' = x^5$$

bo‘ladi.

8.4. Aniqmas integralning sodda xossalari

1°. $f(x)$ funksiya aniqmas integrali $\int f(x)dx$ ning differensiali $f(x)dx$ da teng, ya’ni

$$d\left(\int f(x)dx\right) = f(x)dx .$$

2°. Funksiya differensialining aniqmas integrali shu funksiya bilan o‘zgarmas son yig‘indisiga teng, ya’ni

$$\int dF(x) = F(x) + C \qquad (C = const).$$

8.5. Integralashning sodda qoidalari

1°. Agar $f(x)$ va $g(x)$ funksiyalar boshlang‘ich funksiyalarga ega bo‘lsa, u holda $f(x) + g(x)$ ham boshlang‘ich funksiyaga ega va

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

formula o‘rinli (***integralning additivlik xossasi***).

2°. Agar $f(x)$ funksiya boshlang‘ich funksiyaga ega bo‘lsa, u holda $k \cdot f(x)$ (k-o‘zgarmas son) ham boshlang‘ich funksiyaga ega va $k \neq 0$ da

$$\int k \cdot f(x)dx = k\int f(x)dx$$

formula o‘rinli.

8.6. Elementar funksiyalarning aniqmas integrali

1. $\int 0 \cdot dx = C, \quad C = const .$
2. $\int 1 \cdot dx = \int dx = x + C .$
3. $\int x^{\mu} dx = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1) \quad \left(\int (ax+b)^{\mu} dx = \frac{1}{a}\frac{(ax+b)^{\mu+1}}{\mu+1} + C\right).$

4. $\int \frac{1}{x}dx = \int \frac{dx}{x} = \ln|x| + C \quad (x \neq 0) \quad \left(\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + C\right)$.

5. $\int \frac{1}{1+x^2}dx = \int \frac{dx}{1+x^2} = arctgx + C \quad \left(-\int \frac{dx}{1+x^2} = arcctgx + C\right)$.

6. $\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \quad \left(-\int \frac{dx}{\sqrt{1-x^2}} = \arccos x + C\right)$.

7. $\int a^x dx = \frac{a^x}{\ln a} + C \quad \left(\int e^x dx = e^x + C\right)$.

8. $\int \sin x dx = -\cos x + C \quad \left(\int \sin ax dx = -\frac{1}{a}\cos ax + C\right)$.

9. $\int \cos x dx = \sin x + C \quad \left(\int \cos ax dx = \frac{1}{a}\sin ax + C\right)$.

10. $\int \frac{1}{\sin^2 x}dx = \int \frac{dx}{\sin^2 x} = -ctgx + C \quad \left(\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}ctgax + C\right)$.

11. $\int \frac{1}{\cos^2 x}dx = \int \frac{dx}{\cos^2 x} = tgx + C \quad \left(\int \frac{dx}{\cos^2 ax} = \frac{1}{a}tgax + C\right)$.

12. $\int shxdx = chx + C$.

13. $\int chxdx = shx + C$.

14. $\int \frac{1}{sh^2 x}dx = \int \frac{dx}{sh^2 x} = -cthx + C$.

15. $\int \frac{1}{ch^2 x}dx = \int \frac{dx}{ch^2 x} = thx + C$.

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}arctg\frac{x}{a} + C;$

17. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C; \ (a \neq 0)$.

18. $\int \frac{xdx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln\left|a^2 \pm x^2\right| + C;$

19. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C; \ (a > 0)$.

20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C; \ (a > 0)$.

21. $\int \frac{xdx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C;$

22. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C; \ (a > 0)$.

23. $\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$.

8.7. Integrallash usullari

1. ***O'zgaruvchilarni almashtirib integrallash***. Ushbu $\int f(x)dx$ aniqmas integralni hisoblash talab etilgan bo'lsin. Bunda $f(x)$ funksiya biror $X=(a,b)$ intervalda aniqlangan va $f(x)=\varphi(g(x))g'(x)$ ko'rinishida yozish mumkin deylik.

Agar $\varphi(t)$ funksiya $T=(t_1,t_2)$ intervalda boshlang'ich funksiya $\Phi(t)$ ga ega bo'lib, $g(x)$ funksiya $X=(a,b)$ intervalda (bunda $g(x)\in T$) differensialanuvchi bo'lsa, u holda

$$\int f(x)dx=\int\varphi(g(x))g'(x)dx=\Phi(g(x))+C$$

formula o'rinli.

Misol. Ushbu

$$\int\frac{arctgx}{1+x^2}dx$$

aniqmas integralni hisoblang.

$d(arctgx)=\dfrac{dx}{1+x^2}$ ekanligini e'tiborga olsak,

$$I=\int\frac{arctgx}{1+x^2}dx=\int arctgx\,d(arctgx).$$

Bundan ko'rinadiki, $arctgx=t$ desak, t orqali integral quyidagi integralga keladi:

$$I=\int tdt,$$

u holda

$$I=\frac{t^2}{2}+C.$$

Birinchi integralning x o'zgaruvchi orqali ifodasi

$$I=\frac{1}{2}arctg^2x+C$$

ko'rinishda bo'ladi.

2. ***Bo'laklab integrallash usuli.*** Ikkita $u=u(x)$ va $v=v(x)$ funksiya (a,b) intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. Ma'lumki,

$$d[u(x)\cdot v(x)]=u(x)\cdot dv(x)+v(x)\cdot du(x).$$

Bundan

$$u(x)dv(x)=d[u(x)\cdot v(x)]-v(x)du(x), \qquad (1)$$

ekanligi ravshan.

Endi (1) tenglikni integrallab topamiz:

$$\int u(x)dv(x)=\int[d[u(x)\cdot v(x)]-v(x)du(x)]=u(x)\cdot v(x)-\int v(x)du(x).$$

Shunday qilib, quyidagi

$$\int u(x)dv = u(x)\cdot v(x) - \int v(x)du$$

formulaga kelamiz. Bu formula bo‘laklab integrallash formulasi deyiladi.

Misol. Ushbu

$$I = \int x\sin xdx$$

aniqmas integralni hisoblang.

Integral ostidagi ifodani $u = x$, $dv = \sin xdx$ lar ko‘raytmasi deb olamiz. U holda $du = dx$, $v = -\cos x$ bo‘ladi. Bo‘laklab integrallash formulasiga ko‘ra:

$$I = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C\,.$$

Eslatma. $\int u(x)dv(x)$ integralni hisoblashda logarifmik funksiya, ko‘phad, teskari trigonometrik funksiyalarni $u(x)$ deb belgilash maqsadga muvofiq.

8.8. Ko‘phad va uning ildizlari haqida

Biror

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \qquad (2)$$

ko‘phad berilgan bo‘lsin, bunda $a_0, a_1, a_2, \ldots,$ a_n-o‘zgarmas haqiqiy sonlar, $a_n \neq 0$, $n \in N$ esa ko‘phadning darajasi.

$\alpha \in R$ son uchun $P(\alpha) = 0$ bo‘lsa, α son $P(x)$ ko‘phadning ildizi deyiladi. U holda Bezu teoremasiga ko‘ra, $P(x)$ ko‘phad $x - \alpha$ ga qoldiqsiz bo‘linib, u quyidagi

$$P(x) = (x-\alpha)Q(x)$$

ko‘rinishda ifodalanadi, bunda $Q(x)-$ $(n-1)$-darajali ko‘phad.

Agar (2) ko‘phad $(x-\alpha)^k$ ($k \in N$) ga qoldiqsiz bo‘linsa, α son (2) ko‘phadning k karrali ildizi bo‘ladi. Bu holda $P(x)$ ko‘phadni ushbu $P(x) = (x-\alpha)^k R(x)$ ko‘rinishida ifodalash mumkin, bunda $R(x)-$ $(n-k)$-darajali ko‘phad.

Agar $h = \alpha + i\beta$ kompleks son $P(x)$ ko‘phadning ildizi bo‘lsa, u holda $\bar{h} = \alpha - i\beta$ kompleks son ham shu ko‘phadning ildizi bo‘ladi. Shuningdek, $h = \alpha + i\beta$ son $P(x)$ ning k karrali ildizi bo‘lsa, $\bar{h} = \alpha - i\beta$ son ham k karrali ildizi bo‘ladi.

Teorema. Har qanday n-darajali

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

ko‘phad ($a_0, a_1, a_2, \ldots,$ a_n-o‘zgarmas haqiqiy sonlar, $a_n \neq 0$), ushbu

$$P(x)=(x-\alpha_1)^{\lambda_1}(x-\alpha_2)^{\lambda_2}....(x-\alpha_k)^{\lambda_k}(x^2+p_1x+$$
$$+q_1)^{\gamma_1}(x^2+p_2x+q_2)^{\gamma_2}...(x^2+p_sx+q_s)^{\gamma_s}$$

ko‘rinishda ifodalash mumkin, bunda

$$\lambda_1+\lambda_2+...+\lambda_k+2(\gamma_1+\gamma_2+....+\gamma_s)=n$$

bo‘lib, $(x^2+p_jx+q_j)=0\ (j=1,2,3,...,s)$ tenglamalar haqiqiy ildizga ega emas.

8.9. Sodda kasr

Ushbu

$$\frac{A}{(x-a)^m},\frac{Bx+C}{(x^2+px+q)^m},\ m=1,2,3,...$$

ko‘rinishidagi kasrlarga sodda kasrlar deyiladi, bunda A, B, C hamda a, p, q lar o‘zgarmas sonlar, x^2+px+q kvadrat uchhad esa haqiqiy ildizga ega emas.

8.10. Ratsional funksiya va to‘g‘ri kasr

Quyidagi

$$P(x)=a_0+a_1x+a_2x^2+...+a_nx^n$$

va

$$Q(x)=b_0+b_1x+b_2x^2+...+b_vx^v$$

ko‘phadlarning ($a_0, a_1, a_2,...,\ a_n, b_0, b_1, b_2,...,\ b_v$ -o‘zgarmas sonlar $n\in N, v\in N$) nisbati

$$\frac{P(x)}{Q(x)}=\frac{a_0+a_1x+a_2x^2+...+a_nx^n}{b_0+b_1x+b_2x^2+...+b_vx^v}$$

kasr ***ratsional funksiya*** deyiladi, $n<v$ bo‘lsa u ***to‘g‘ri kasrga*** aylanadi.

Teorema. Har qanday to‘g‘ri kasr sodda kasrlar yig‘indisi orqali ifodalanadi.

8.11. Sodda kasrlarni integrallash

Sodda kasrlarni aniqmas integralini topish.

1°. $\frac{A}{x-a}$ sodda kasrning aniqmas integrali:

$$\int\frac{A}{x-a}dx=A\int\frac{d(x-a)}{x-a}=A\ln|x-a|+C.$$

2°. $\frac{A}{(x-a)^m}$ $(m>1)$ sodda kasrning aniqmas integrali:

$$\int\frac{A}{(x-a)^m}dx=A\int\frac{d(x-a)}{(x-a)^m}=A\int(x-a)^{-m}d(x-a)=\frac{A}{1-m}\cdot\frac{1}{(x-a)^{m-1}}+C.$$

3°. $\dfrac{Bx+C}{x^2+px+q}$ sodda kasrning integrali $I=\int\dfrac{Bx+C}{x^2+px+q}dx$ ni hisoblash uchun avval kasrning maxrajida turgan x^2+px+q kvadrat uchhadni ushbu

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}$$

ko‘rinishida yozib olamiz. U holda

$$I=\int\frac{Bx+C}{x^2+px+q}dx=\int\frac{Bx+C}{\left(x+\frac{p}{2}\right)^2+a^2}dx$$

bo‘ladi, bunda $a^2=q-\dfrac{p^2}{4}$. Bu integralda $x+\dfrac{p}{2}=t$ almashtirish bajaramiz:

$$I=B\int\frac{tdt}{t^2+a^2}+\left(C-\frac{Bp}{2}\right)\int\frac{dt}{t^2+a^2}=\frac{B}{2}\int\frac{d(t^2+a^2)}{t^2+a^2}+\left(C-\frac{Bp}{2}\right)\frac{1}{a}\int\frac{d\left(\frac{t}{a}\right)}{1+\left(\frac{t}{a}\right)^2}=$$

$$=\frac{B}{2}\ln\left[t^2+a^2\right]+\left(C-\frac{Bp}{2}\right)\frac{1}{a}arctg\frac{t}{a}+C_*=\frac{B}{2}\ln\left[x^2+px+q\right]+\frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C_2\,.$$

Demak,

$$\int\frac{Bx+C}{x^2+px+q}dx=\frac{B}{2}\ln\left[x^2+px+q\right]+\frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C_2\,,$$

bunda C_2-ixtiyoriy o‘zgarmas.

4°. $\dfrac{Bx+C}{(x^2+px+q)^m}$ sodda kasrning integrali $I_m=\int\dfrac{Bx+C}{(x^2+px+q)^m}dx$ ni hisoblash uchun 3°-holdagidek o‘zgaruvchilarni almashtiramiz: $x+\dfrac{p}{2}=t$. Natijada quyidagiga ega bo‘lamiz:

$$I_m=\int\frac{Bx+C}{(x^2+px+q)^m}dx=\int\frac{Bx+C}{\left[\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right]^m}dx=$$

$$=\int\frac{Bt+\left(C-\frac{Bp}{2}\right)}{(t^2+a^2)^m}dt=\frac{B}{2}\int\frac{d(t^2+a^2)}{(t^2+a^2)^m}+\left(C-\frac{Bp}{2}\right)\int\frac{dt}{(t^2+a^2)^m}=$$

$$=\frac{B}{2}\cdot\frac{1}{1-m}\cdot\frac{1}{(t^2+a^2)^{m-1}}+\left(C-\frac{Bp}{2}\right)\int\frac{dt}{(t^2+a^2)^m}\,.$$

Bunda $\int \frac{dt}{(t^2+a^2)^m}$ integral rekurrent formula orqali topiladi.

8.12. Ratsional funksiyalarning aniqmas integrali

$f(x)$ ratsional funksiya bo‘lib, uning aniqmas integralini hisoblash talab etilsin.

Ma’lumki, ratsional funksiya ikkita $P(x)$ va $Q(x)$-butun ratsional funksiyalarning nisbatidan iborat, ya’ni

$$f(x)=\frac{P(x)}{Q(x)}.$$

Agar $\frac{P(x)}{Q(x)}$ noto‘g‘ri kasr (suratidagi ko‘phadning darajasi maxrajdagi ko‘phadning darajasidan katta) bo‘lsa, uning butun qismini ajratib, butun ratsional funsiya hamda to‘g‘ri kasr yig‘indisi ko‘rinishida quyidagicha ifodalab olinadi:

$$\frac{P(x)}{Q(x)}=R(x)+\frac{P_1(x)}{Q(x)}.$$

U holda

$$\int f(x)dx=\int\frac{P(x)}{Q(x)}dx=\int R(x)dx+\int\frac{P_1(x)}{Q(x)}dx$$

bo‘ladi.

8.13. Ba’zi irratsional ko‘rinishidagi funksiyalarni integrallash. Eyler almashtirishlari

$R(x,y)$ deganda x va y o‘zgaruvchiga nisbatan ratsional bo‘lgan funksiyani tushunamiz.

I) $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ integralni topishda $t=\sqrt[n]{\frac{ax+b}{cx+d}}$ almashtirish bajarilsa, ratsional funksiyani integrallashga keladi.

II) $\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{r_1},\left(\frac{ax+b}{cx+d}\right)^{r_2},\ldots,\left(\frac{ax+b}{cx+d}\right)^{r_n}\right)dx$ integralni topishda $r_1,r_2,\ldots,r_n$ ratsional sonlarning umumiy m maxrajga keltirib, integralda $t=\sqrt[m]{\frac{ax+b}{cx+d}}$ almahtirish bajarilsa, ratsional funksiyani integrallashga keladi.

III) $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ integralni topishda quyidagi 3 ta hol qaraladi.

1 – hol. ax^2+bx+c kvadrat uchhad har xil x_1 va x_2 haqiqiy ildizlarga ega bo'lsin. Bundan $ax^2+bx+c=a(x-x_1)(x-x_2)$.

$$\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1)$$

almashtirish bajaramiz.

2 - hol. $a>0$ bo'lsin. Unda

$$\sqrt{ax^2+bx+c)}=t-\sqrt{a}x\ (yoki\sqrt{ax^2+bx+c}=t+\sqrt{a}x)$$

almashtirish bajaramiz.

3 - hol. $c>0$ bo'lsin. U holda

$$\sqrt{ax^2+bx+c}=tx+\sqrt{c}\ (yoki\sqrt{ax^2+bx+c}=tx-\sqrt{c})$$

almashtirishni bajarish yordamida topilishi kerak bo'lgan integral ratsional funksiyani integrallashga keltiriladi.
Yuqorida keltirilgan almashtirishlarga ***Eyler almashtirishlari*** deb aytiladi.

8.14. Binomial differensiallarni integrallash

a*) Ta'rif.** Ushbu $x^m(a+bx^n)^p dx$ ko'rinishidagi ifodaga ***binomial differensial deb ataladi. Bu yerda m,n,p - lar ratsional sonlar.

$$I=\int x^m(a+bx^n)^p dx$$

integral quyidagi 3 ta holda ratsional funksiyaning integraliga keladi.

1 - hol. p - *butun son*. $x=t^N$ almashtirish bajariladi. Bu erda N soni m va n ratsional sonlar (ya'ni kasrlar) maxrajlarning eng kichik umumiy bo'linuvchisi.

2 - hol. $\frac{m+1}{n}$ - *butun son*. Bu holda $a+bx^n=Z^N$ almashtirish bajarish kerak bo'ladi. $N-p$ ratsional sonning maxraji.

3 - hol. $\frac{m+1}{n}+p$ - *butun son*. Bunda $\frac{a}{x^n}+b=Z^N$, $N-p$ ning maxaraji, almashtirish bajarish etarli.

Ushbu $\int R(\sin x,\cos x)dx$ ko'rinishidagi integral, bunda R ratsional funksiya umumiy holda $tg\frac{x}{2}=t$ $(-\pi<x<\pi)$ almashtirish bilan t o'zgaruvchining ratsional funksiyasiga aytiladi. Bu unversal almashtirishda $\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, $dx=\frac{2dt}{1+t^2}$ bo'ladi.

1) agar $R(\sin x,\cos x)$ ratsional funksiya $\sin x$ ni $-\sin x$ almashtirishda $R(-\sin x,\cos x)=-R(\sin x,\cos x)$ tenglik o'rinli bo'lsa, $\cos x=t$ deb belgilash ma'qul;

2) agar $R(\sin x, \cos x)$ ratsional funksiyada $\cos x$ ni $-\cos x$ ga almashtirilganda $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ tenglik o'rinli bo'lsa $\sin x = t$ deb belgilash ma'qul;

3) agar $\sin x$ ni $-\sin x$ bilan va $\cos x$ ni $-\cos x$ bilan almashtirishda $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ o'rinli bo'lsa, $tgx = t$ deb belgilash ma'qul;

4) Aytaylik,

$$I = \int \sin^n x \cdot \cos^m x dx, \ (n, m \in Z)$$

integral berilgan bo'lsin. Bu integralni topish uchun quyidagi hollar qaraladi.

1 - hol. n - toq, m - juft, bunda $\cos x = t$ almashtirish bajariladi.

2 - hol. n - juft, m - toq, bunda $\sin x = t$ almashtirish bajariladi.

3 - hol. n va m - toq. Bunda $\cos x = t$, $\sin x = t$ yoki $tgx = t$ almashtirishlardan biri bajariladi.

4 - hol. n va m - juft. Bu holda

$$\sin 2x = 2\sin x \cdot \cos x \text{ va } \cos 2x = \cos^2 x - \sin^2 x$$

formulalardan foydalanib tartib pasaytiriladi va yuqoridagi hollardan biriga keltiriladi.

M5. Aniqmas integrallarga doir mashqlar

1. Integrallar jadvalidan va aniqmas integralning xossalaridan foydalanib integrallang

1) $\int(6x^3 + \frac{1}{x} + e^x + a^x)dx$ 2) $\int\left(\sqrt{x} + \sqrt[3]{x} + 3\sqrt[4]{x} - 1{,}2\right)dx$

3) $\int\left(\frac{1}{x^2} + \frac{3}{x^3} + \frac{4}{x^4} - 6{,}7\right)dx$ 4) $\int 2^x\, 3^x\, dx$

5) $\int \frac{(1+x)^3}{3x^2}dx$ 6) $\int \frac{dx}{\sqrt{5-5x^2}}dx$

7) $\int \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$ 8) $\int \frac{2^{3x}+1}{2^x+1}dx$

9) $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx$ 10) $\int \frac{cos2x}{\cos^2 x \cdot \sin^2 x}dx$

11) $\int \mathrm{ctg}^2\, xdx$ 12) $\int(arcsin5x + arccos5x)d\,x$

13) $\int \frac{x^4}{\sqrt{3+x^5}}dx$ 14) $\int x^3\sqrt{1-x^4}dx$

15) $\int \frac{dx}{6+4x^2}$ 16) $\int \frac{dx}{6-4x^2}$

17) $\int \frac{dx}{\sqrt{6-4x^2}}$ 18) $\int \frac{cosx}{1+sinx}dx$

19) $\int \frac{arctgx}{1+x^2}dx$ 20) $\int \frac{sinxdx}{\sqrt[3]{\cos^2 x}}$

21) $\int 2^{sinx} cosxdx$ 22) $\int \frac{x^5 dx}{\sqrt{5+x^5}}dx$

2. Integral ostidagi ifodaning ko'rinishini o'zgartirib quidagi integrallarni hisoblang.

1) $\int \frac{5x}{x+5}dx$ 2) $\int \frac{12x}{12+6x}dx$

3) $\int \frac{3x-1}{x-3}\mathrm{dx}$ 4) $\int \frac{(1+x)^2}{x^3+1}dx$

5) $\int cos\frac{1}{x}\frac{dx}{x^2}$ 6) $\int \frac{dx}{x\sqrt{x^2-1}}$

7) $\int \frac{dx}{\sqrt{x(1-x)}}$ 8) $\int \frac{dx}{\sqrt{x(1+x)}}$

9) $\int \frac{sinx}{\sqrt{\cos^2 x}}dx$ 10) $\int \frac{dx}{sinx}$

11) $\int \frac{x^4}{x+1}dx$ 12) $\int \frac{dx}{(x+1)(2x-3)}$

13) $\int \frac{dx}{x^2-7x+12}$ 14) $\int \frac{dx}{(x-2)^2+4}\mathrm{dx}$

15) $\int \sin^3 xdx$ 16) $\int \frac{dx}{\sin^2 x\, cosx}$

3. Integrallashning asosiy usullaridan foydalanib integrallang.

1) $\int \frac{dx}{1+\sqrt{x+1}}$ 2) $\int \frac{x^2dx}{\sqrt{x-1}}$

3) $\int \frac{4x+3}{(x-2)^2}dx$ 4) $\int \frac{dx}{x\sqrt{x+1}}$

5) $\int \frac{dx}{1+\sqrt{x}}$ 6) $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

7) $\int \frac{\sqrt{1+lnx}}{x\, lnx}dx$ 8)$\int \sin^7 xdx$

9)$\int x\, sin\sqrt{x}\, dx$

4.Trigonametrik yoki giperbolik almashtirishlardan foydalanib quyidagi inegrallarni hisoblang.

1) $\int \frac{dx}{x^2\sqrt{x^2+4}}$ 2) $\int \frac{x^2dx}{\sqrt{9-x^2}}$

3) $\int \sqrt{1-x^2}\, dx$ 4) $\int \sqrt{9+x^4}\, dx$

5) $\int \frac{dx}{\sqrt[3]{(1-x^2)^2}}$ 6) $\int \sqrt{\frac{9+x}{9-x}}dx$

7) $\int \sqrt{16+x^2}\, dx$ 8) $\int \frac{x^2}{\sqrt{25+x^2}}dx$

5. Bo'laklab integrallash usulidan foydalanib quyidagi integrallarni hisoblang.

1) $\int x\, cos2x\, dx$ 2) $\int x5^x dx$

3) $\int x^3 lnxdx$ 4) $\int x\, arcctgxdx$

5) $\int arcsinxdx$ 6) $\int x^3 e^{-x^2}dx$

7) $\int \frac{arccosx}{x^2}dx$ 8) $\int (arccosx)^2 dx$

9) $\int \cos(lnx)\, dx$ 10) $\int \sqrt{3-x^2}\, dx$

11) $\int x\, sin\sqrt{x}\, dx$ 12) $\int \frac{x^2dx}{1-x^4}$

6. Quyidagi ratsional funksiyalarni integrallang.

1) $\int \frac{dx}{x(x-1)(x+1)}$ 2) $\int \frac{xdx}{(x-2)(x-1)(x+1)}$

3) $\int \frac{2x^2-5}{x^4-5x^2+6}dx$ 4) $\int \frac{dx}{(x+1)^2(x+2)^3}$

5) $\int \frac{xdx}{(x+1)(x^2+1)}$ 6) $\int \frac{2x^2-x^5}{1+x^6}dx$

7) $\int \frac{x^2}{1-x^4}dx$ 8) $\int \frac{xdx}{(x+2)(2x+1)}$

7. Quyidagi trigonometrik funksiyalarni integrallang.

1) $\int sinx\ sin5xdx$ 2) $\int cos2x\ cos4xdx$

3) $\int sin2x\ sin4x\ sin6xdx$ 4) $\int cos2x\ cos4x\ cos6xdx$

5) $\int \cos^2 2x\ \cos^2 3xdx$ 6) $\int \frac{sin2x}{\cos^3 x}dx$

7) $\int \cos^3 xdx$ 8) $\int sin5x\ sin7x\ sin9xdx$

9) $\int \sin^2 2x\ \cos^2 2xdx$ 10) $\int \frac{dx}{(sinx+cosx)^2}$

11) $\int \frac{dx}{1+tgx}$ 12) $\int \frac{2-sinx}{2+cosx}dx$

13) $\int \frac{cosxdx}{\sin^3 x-\cos^3 x}$ 14) $\int \frac{dx}{\sin^4 x+\cos^4 x}$

15) $\int \sin^4 x\ \cos^6 x\ dx$

8. Quyidagi binomial differensiallarning integralini hisoblang.

1) $\int \frac{dx}{\sqrt[4]{1+x^4}}$ 2) $\int \frac{dx}{x\sqrt[3]{x^2+1}}$

3) $\int \frac{dx}{\sqrt[3]{1+x^3}}$ 4) $\int (\sqrt{x}(1+\sqrt[3]{x})^4 dx$

5) $\int x^{-1}(1+x^{\frac{1}{3}})^{-3}dx$ 6) $\int \frac{dx}{\sqrt[3]{1+x^3}}$

7) $\int \frac{\sqrt{1-x^4}}{x^5}dx$ 8) $\int \frac{\sqrt[3]{1+x^3}}{x^2}dx$

9) $\int \frac{dx}{x\sqrt[3]{1+x^5}}$

9-§. ANIQ INTEGRAL

$f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin. Bu kesmani

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

nuqtalar bilan n ta qismga bo'lamiz. Har bir (x_{i-1}, x_i) oraliqdan ixtiyoriy ξ_i nuqtani olamiz va ushbu yig'indini tuzamiz: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, bunda $\Delta x_i = x_i - x_{i-1}$. Ushbu

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

ko'rinishdagi yig'indiga ***integral yig'indi*** (***Riman yig'indisi***) deyiladi.

Agar $\max \Delta x_i \to 0$ da $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ yig‘indining limiti mavjud va chekli bo‘lsa, u holda bu limitga $f(x)$ funksiyadan a dan b gacha olingan **aniq integral** deyiladi va $\int_a^b f(x)dx = \lim_{\max\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ ko‘rinishda belgilanadi. Bu holda $f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi funksiya deyiladi. a va b sonlar mos ravishda integrallashning quyi va yuqori chegaralari deyiladi.

Teorema. Agar $f(x)$ funksiya $[a;b]$ oraliqda uzluksiz bo‘lsa, u shu oraliqda integrallanuvchi bo‘ladi.

Teorema. Agar $f(x)$ funksiya $[a;b]$ oraliqda chegaralangan va monoton bo‘lsa, u shu oraliqda integrallanuvchi bo‘ladi.

9.1. Aniq integral xossalari

1⁰. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

2⁰. $\int_a^a f(x)dx = 0$.

3⁰. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

4⁰. $\int_a^b (f_1(x) \pm f_2(x))dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$.

5⁰. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, bunda k - o‘zgarmas.

6⁰. Agar $[a;b]$ kesmada $f(x) \geq 0$ bo‘lsa, u holda $\int_a^b f(x)dx \geq 0$.

7⁰. Agar $[a;b]$ kesmada $f(x) \geq g(x)$ bo‘lsa, u holda $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

8⁰. Agar m va M mos ravishda $f(x)$ funksiyaning $[a;b]$ kesmadagi eng kichik va eng katta qiymati bo‘lsa, u holda $m \cdot (b-a) \leq \int_a^b f(x)dx \leq M(b-a)$ tengsizlik o‘rinli (aniq integralni baholash haqidagi teorema).

9⁰. $\int_a^b f(x)dx = f(c)(b-a)$ bunda $c \in (a;b)$ (o‘rta qiymat haqidagi teorema.)

10⁰. Koshi-Bunyakovskiy tengsizligi. $f(x)$ va $g(x)$ funksiyalar $[a;b]$ segmentda integrallanuvchi bo‘lsa, u holda

$$\int_a^b |f(x)\cdot g(x)|dx \le \sqrt{\int_a^b f^2(x)dx}\cdot\sqrt{\int_a^b g^2(x)dx}\,.$$

9.2. Aniq integralni hisoblash

Agar $F(x)$ funksiya $[a;b]$ kesmada uzluksiz $f(x)$ funksiyaning boshlang'ich funksiyalaridan biri bo'lsa, u holda quyidagi formula o'rinli:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a),$$

bu formula ***Nyuton –Leybnis formulasi*** deyiladi.

O'zgaruvchini almashtirib integrallash formulasi. Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz, $x=\varphi(t)$ funksiya esa $t\in[\alpha,\beta]$ da aniqlangan va uzluksiz $a=\varphi(\alpha),\ b=\varphi(\beta)$ bo'lsa, hamda uzluksiz hosilaga ega bo'lsa,

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

tenglik o'rinli.

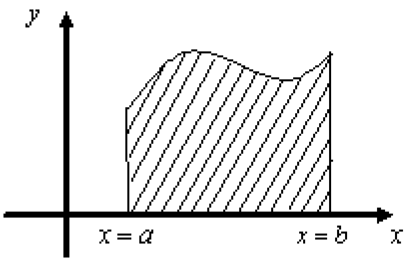
Bo'laklab integrallash formulasi. Agar $u=u(x),\ v=v(x)$ funksiyalar va ularning hosilalari $[a;b]$ kesmada uzluksiz bo'lsa, u holda

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

tenglik o'rinli.

9.3. Aniq integralning tatbiqlari. Shaklning yuzasini topish

1) $f(x)\in C[a,b]$ ($C[a,b]$ $[a,b]$ dagi uzluksiz funksiyalar sinfi) bo'lib, ixtiyoriy $x\in[a,b]$ uchun $f(x)\ge 0$ tengsizlik bajarilsin. Yuqoridan $f(x)$ funksiya grafigi, yon tomondan $x=a$ va $x=b$ vertikal chiziqlar va Ox bilan chegaralangan egri chiziqli trapetsiya (soha yuzi)



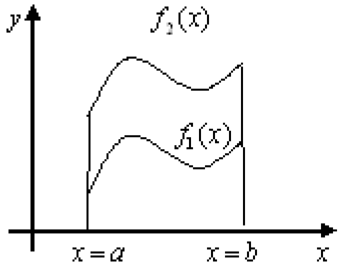
24-rasm

$$S=\int_a^b f(x)\,dx$$

bo‘ladi.

2) Agar tekislikda shakl $f_1(x)\in C[a,b]$, $f_2(x)\in C[a,b]$ (ixtiyoriy $x\in[a,b]$ da $0\le f_1(x)\le f_2(x)$) $x=a$ va $x=b$ chiziqlar bilan chegaralangan bo‘lsa, uning yuzi quyidagi aniq integral bilan topiladi

$$S=\int_a^b [f_2(x)-f_1(x)]dx.$$

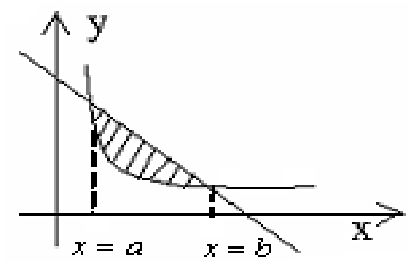


25-rasm

Misol. Ushbu $xy=4;\ x+y=5$ shakl yuzi topilsin.

Yechish: $xy=4,\ x+y=5,\quad y=\frac{4}{x},\ y=5-x$.

Soha 26-rasmda shtrixlangan chiziqlarning kesishish nuqtalarini topamiz.



26-rasm

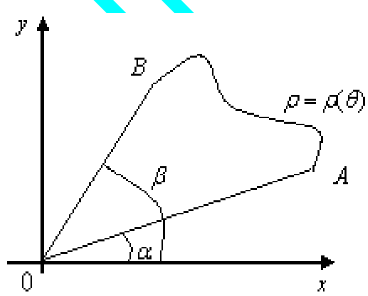
$\frac{4}{x}=5-x$, $x(5-x)=4,\ 5x-x^2-4=0,\ x^2-5x+4=0$, $x_1=\frac{5+3}{2}=4,\ x_2=\frac{5-3}{2}=1$ ($x_1=b, x_2=a$), $S=\int_1^4\left(5-x-\frac{4}{x}\right)dx=\left(5x-\frac{x^2}{2}+\frac{4}{x^2}\right)\Big|_1^4=20-8+\frac{1}{4}-5+\frac{1}{2}-4=3+\frac{3}{4}=\frac{15}{4}$.

9.4. Qutb koordinatalar sistemasida berilgan shaklning yuzasini aniq integral yordamida topish

Qutb koordinatalar sistemasida $\rho=\rho(\theta)$ ($\alpha\le\theta\le\beta$) funksiyaning yoyi $\overset{\cup}{AB}$ hamda OA va OB radius vektorlar bilan chegaralangan shakl (egri chiziqli sohalar) yuzi

$$S=\frac{1}{2}\int_\alpha^\beta \rho^2(\theta)d\theta$$

formula bilan topiladi. Bunda $\rho(\theta)$ funksiya $[\alpha,\beta]$ oraliqda uzluksiz funksiya va $\rho(\theta)\ge 0$.



27-rasm

Tekislikda

$$\begin{cases} x=x(t)\\ y=y(t)\end{cases}\qquad (\alpha\le t<\beta)$$

parametrik tenglamalar bilan chegarlangan shakl (egri chiziqli soha) berilgan bo‘lsin.

a) $x = x(t)$, $y = y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz, ixtiyoriy $t \in [\alpha,\beta]$ da $x(t) \geq 0$, $y(t) \geq 0$ va $x'(t)$ uzluksiz va $x'(t) \geq 0$ bo‘lsa, shakl yuzi

$$S = \int_{\alpha}^{\beta} y(t) \cdot x'(t)dt$$

formula bilan topiladi.

b) $x = x(t)$ funksiya $[\alpha,\beta]$ da uzluksiz, ixtiyoriy $t \in [\alpha,\beta]$ da $x(t) \geq 0$, $y(t) \geq 0$ va $y(t)$ funksiya uzluksiz va $y'(t) \geq 0$ hosilaga ega bo‘lsa, u holda shakl yuzi

$$S = \int_{\alpha}^{\beta} x(t) \cdot y'(t)dt$$

bo‘ladi.

9.5. Aniq integral yordamida yoy uzunligini topish

1) $f(x)$ funksiya $[a,b]$ kesmada aniqlangan uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo‘lsin. Shu funksiya grafigidadagi $A(a, f(a))$ va $B(b, f(b))$ nuqtalar orasidagi $\overset{\cup}{AB}$ egri chiziq yoyi uzunligi

$$l = \int_{a}^{b} \sqrt{1+[f'(x)]^2}\, dx$$

formula bilan topiladi.

Agar $b = x$ desak, $l(x) = \int_{a}^{x} \sqrt{1+[f'(x)]^2}\, dx$ bo‘lib,

$$\frac{dl}{dx} = \sqrt{1+[f'(x)]^2} \Rightarrow dl = \sqrt{1+[f'(x)]^2}\, dx$$

Bu ifodaga ***yoy differensiali*** deb ataladi.

2) ***Parametrik ko‘rinishda berilgan egri chiziq yoyining uzunligini topish***.

Agar

$$\overset{\cup}{AB}: \begin{cases} x = \varphi(t) \\ y = \psi(t), \end{cases} \quad \alpha \leq t \leq \beta$$

bo‘lib, $\varphi'(t) \in C[\alpha,\beta]$ va $\psi'(t) \in C[\alpha,\beta]$ bo‘lsa,

$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'(t)^2 + [\psi'(t)]^2}\, dt$$

bo‘ladi.

3) Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligini topish.

3.1. Agar

$$\overset{\cup}{AB}:\begin{cases}\alpha\le\varphi\le\beta,\\ r=r(\varphi)\end{cases}$$

bo‘lib, $r'(\varphi)\in C[\alpha,\beta]$ bo‘lsa, unda

$$l=\int_{\alpha}^{\beta}\sqrt{r^2(\varphi)+[r'(\varphi)]^2}\,d\varphi$$

formula o‘rinli bo‘ladi.

Misol. $r=4e^{\frac{5\varphi}{4}}$ $\quad \frac{\pi}{2}\le\varphi\le\pi$

Yechish: Egri chiziq yoyi uzunligi $\ell=\int_{\alpha}^{\beta}\sqrt{r'^2(\varphi)+r^2(\varphi)}d\varphi$ formula orqali hisoblanadi.

$$r'=4\cdot\frac{5}{4}e^{\frac{5\varphi}{4}}=5e^{\frac{5\varphi}{4}},\ \ \ell=\int_{\frac{\pi}{2}}^{\pi}\sqrt{25e^{\frac{5\varphi}{2}}+16e^{\frac{5\varphi}{2}}}\,d\varphi=\int_{\frac{\pi}{2}}^{\pi}e^{\frac{5\varphi}{4}}\sqrt{41}d\varphi=\sqrt{41}\cdot\frac{4}{5}e^{\frac{5\varphi}{4}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4\sqrt{41}}{5}\left(e^{\frac{5\pi}{4}}-e^{\frac{5\pi}{8}}\right).$$

3.2. Agar

$$\overset{\cup}{AB}:\begin{cases}\rho_1\le\rho\le\rho_2\\ \varphi=\varphi(\rho)\end{cases}$$

bo‘lsa, yoy uzunligi

$$l=\int_{\rho_1}^{\rho_2}\sqrt{[\rho\varphi'(\rho)]^2+1}\,d\rho$$

formula bilan topiladi.

Misol. Parametr ko‘rinishida berilgan sikloida yoyining uzunligini toping: $x=a(t-\sin t)$, $y=a(1-\cos t)$ $(0\le t\le 2\pi)$.

Yechish:

$$x'(t)=a(1-\cos t),\ \ y'(t)a\sin t;$$

$$x'(t)^2+y'(t)^2=2a^2(1-\cos t)=4a^2\sin^2\frac{t}{2};$$

Yuqoridagi formuladan

$$l=\int_0^{2\pi}\sin\frac{t}{2}dt=4a\int_0^{2\pi}\sin zdz=\left|z=\frac{t}{2}\right|=4a\cos z\Big|_{\pi}^{0}=8a.$$

9.6. Aylanma sirtning yuzasi

1. $f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo‘lsin, uning grafigi quyidagi

$$\{(x,\ f(x)):x\in[a;b]\}$$

nuqtalar to‘plamidan iborat. Shu grafikdagi $A(a,f(a))$ va $B(b,f(b))$ nuqtalar orasidagi $\overset{\cup}{AB}$ egri chiziqni qaraymiz.

Aytaylik, $f(x) \in C[a,b]$ bo'lib, $f(x) \geq 0$ bo'lsin. $\overset{\cup}{AB}$ yoyni OX o'qi atrofida aylantiramiz va aylanma sirtni hosil qilamiz. Agar $f'(x) \in C[a,b]$ bo'lsa, unda shu aylanma sirtning yuzasi ushbu

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1+[f'(x)]^2}\, dx$$

formula yordamida hisoblanadi.

2. $\overset{\cup}{AB}$ egri chiziq yuqori yarim tekislikda joylashgan bo'lib u $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $(\alpha \leq t \leq \beta)$ parametrik tenglama bilan berilgan bo'lsin. Bunda $x = x(t)$ va $y = y(t)$ funksiyalar $[\alpha;\beta]$ da uzluksiz va uzluksiz $x'(t), y'(t)$ hosilalarga ega bo'lsin. Bu egri chiziqni OX o'qi atrofida aylantirishdan hosil bo'lgan aylanma sirtning yuzi

$$S = 2\pi \int_\alpha^\beta y(t)\sqrt{x'^2(t) + y'^2(t)}\,dt$$

formula yordamida hisoblanadi.

9.7. Aniq integral yordamida hajmni hisoblash

1. Faraz qilaylik, bizga biror T jism berilgan bo'lib, uning OY o'qiga parallel bo'lgan kesimlarining yuzasi ma'lum bo'lsin Bu yuza x o'zgaruvchining funksiyasi bo'ladi va uni $S = S(x)$ deb belgilaylik. Agar $S(x) \in C[a,b]$ bo'lsa, unda T jismning hajmi V

$$V = \int_a^b S(x)\,dx$$

formula yordamida hisoblanadi.

2. *Aylanma jismning hajmi*. Ushbu

$$D = \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases}$$

egri chiziqli trapetsiyani OX o'qi atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmi

$$V = \pi \int_a^b [f(x)]^2\, dx$$

formula yordamida hisoblanadi.

Agar shu shaklning o'zi OY o'qi atrofida aylantirilsa, u holda aylanma jismining hajmi

$$V = 2\pi \int_a^b x f(x)\,dx$$

formula bilan hisoblanadi.

3. *Umumiy holda* $a \le x \le b$, $y_1(x) \le y \le y_2(x)$ tekislikni OX o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi

$$V = \pi \int_a^b \left[y_2^2(x) - y_1^2(x) \right] dx$$

formula bilan topiladi.

4. Egri chiziq

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

parametrik tenglamalar bilan berilgan bo‘lsin. Bunda $x = x(t)$ funksiya uzluksiz hamda uzluksiz $x'(t) \ge 0$ hosilaga ega, $y(t)$ funksiya $[\alpha, \beta]$ da uzluksiz hamda $\forall t \in [\alpha, \beta]$ da $y(t) \ge 0$. Bu chiziq bilan chegaralangan shaklning OX o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi

$$V = \pi \int_\alpha^\beta y^2(t) x'(t) dt$$

formula bilan hisoblanadi.

9.8. O‘zgaruvchi kuchning bajargan ishi

OX o‘qida shu o‘q bo‘ylab biror jism $F = F(x)$ kuch ta’sirida harakat qilayotgan bo‘lsin. Agar $F(x) \in C[a,b]$ bo‘lsa, $F = F(x)$ kuch ta’sirida jismni a nuqtadan b nuqtaga o‘tkazishda bajarilgan ish

$$A = \int_a^b F(x)\, dx$$

formula yordamida hisoblanadi.

9.9. Statik moment. Og‘irlik markazi

Aytaylik, m massaga ega bo‘lgan $M(x, y)$ - moddiy nuqta berilgan bo‘lsin. my va mx ko‘paytmalarga mos ravishda berilgan nuqtaning OX **va** OY ***o‘qlarga nisbatan statik momentlari*** deb ataladi.

Egri chiziqning OX va OY o‘qlarga nisbatan ***statik momentlari*** M_x va M_y lar ham shu kabi aniqlanadi hamda

$$M_x = \int_0^l y\, dl, \; M_y = \int_0^l x\, dl$$

formulalar yordamida hisoblanadi. Bu yerda $dl = \sqrt{(dx)^2 + (dy)^2}$ - yoy differensiali, l esa berilgan egri chiziq uzunligi.

Berilgan egri chiziq og‘irlik markazining koordinatalari esa

$$\bar{x} = \frac{M_y}{l}, \; \bar{y} = \frac{M_x}{l}$$

formulalar yordamida hisoblanadi.

9.10. Geometrik shakllarning statik momentlari va og'irlik markazi

Agar geometrik shakl

$$D = \begin{cases} a \le x \le b \\ 0 \le y \le f(x) \end{cases}$$

egri chiziqli trapetsiyadan iborat bo'lsa, unda

$$M_x = \frac{1}{2}\int_a^b y^2\,dx, \quad M_y = \frac{1}{2}\int_a^b xy\,dx$$

va

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{S}, \frac{M_x}{S}\right)$$

bo'ladi. Bu yerda $S = \int_a^b y(x)\,dx$ - trapetsiyaning yuzi.

Misol. Ushbu $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ $(0 \le x \le a,\ 0 \le y \le b)$ sohani Ox, Oy o'qlariga nisbatan statik momentlari va og'irlik markazining koordinatalari topilsin.

Yechish: Ellipsning to'rtdan bir qismi tenglamasini quyidagicha yozib olamiz:

$$\begin{cases} x = a\sin t, \\ y = b\cos t, \end{cases} \quad 0 \le t \le \frac{\pi}{2}$$

Unda

$$M_x = \frac{1}{2}\int_0^{\frac{\pi}{2}} y^2(t)dx(t) = \frac{ab^2}{2}\int_0^{\frac{\pi}{2}} \cos^3 t dt = \frac{ab^2}{3},$$

$$M_y = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(t)y(t)dx(t) = a^2b\int_0^{\frac{\pi}{2}} \sin t\cos^2 t dt = \frac{a^2b}{3}.$$

Agar qaralayotgan soha yuzi $S = \frac{\pi ab}{4}$ ekanligini e'tiborga olsak, unda koordinatalar:

$$\bar{x} = \frac{M_y}{S} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{M_x}{S} = \frac{4b}{3\pi}$$

ga teng bo'ladi.

M6. Aniq integralga doir mashqllar

1. Nyuton –Leybnist formulasidan foydalanib quyidagi integrallarni hisoblang.

1) $\int_0^1 (x^2+x^3+5x^4)dx$ 2) $\int_1^2 \left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\right)dx$

3) $\int_1^2 (\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x})dx$ 4) $\int_0^1 (2e^{2x}+3e^{3x}+4e^{4x})dx$

5) $\int_0^1 (2^x+3^x+4^x)dx$ 6) $\int_0^1 \sqrt{1+x}dx$

7) $\int_{-1}^8 \sqrt[3]{x}dx$ 8) $\int_0^{\pi} (\sin x+\cos x)dx$

9) $\int_0^1 \frac{xdx}{(x^2+1)^2}$ 10) $\int_0^{16} \frac{dx}{\sqrt{x+9}-\sqrt{x}}$

11) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ 12) $\int_0^1 \frac{xdx}{(x^2+1)^2}$

13) $\int_2^3 \frac{dx}{2x^2+3x-2}$ 14) $\int_0^2 |1-x|dx$

15) $\int_{-4}^5 |1+x|dx$ 16) $\int_1^{e^3} \frac{dx}{x\sqrt{1+\ln x}}$

17) $\int_0^{\frac{\pi}{2}} (\sin 2x+\cos 2x)dx$ 18) $\int_0^{\frac{\pi}{4}} \sin^2 2xdx$

2. Aniq integralda bo'laklab integrallash formulasi yordamida quyidagi integrallarni hisoblang.

1) $\int_0^1 xe^x dx$ 2) $\int_0^1 xe^{-x} dx$

3) $\int_0^{\frac{\pi}{2}} x\sin xdx$ 4) $\int_1^2 x^2 \ln xdx$

5) $\int_0^{\pi} x^3 \sin xdx$ 6) $\int_0^{\frac{1}{2}} \arcsin xdx$

7) $\int_0^{\frac{1}{2}} \arccos xdx$ 8) $\int_0^{e} \sin \ln xdx$

9) $\int_1^2 arctg\sqrt{x}dx$ 10) $\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx$

11) $\int_1^3 arctg\sqrt{x}dx$

3. Aniq integralda o'zgaruvchilarni almashtirib, quyidagi integrallarni hisoblang.

1) $\int_0^1 x\sqrt{1+x}dx$ 2) $\int_0^2 \sqrt{4-x^2}dx$

3) $\int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$ 4) $\int_0^{\ln 2} \sqrt{e^x-1}dx$

5) $\int_0^\pi \sin^6\frac{x}{2}dx$ 6) $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{dx}{x^5\sqrt{x^2-1}}$

7) $\int_0^2 \frac{dx}{x+\sqrt{4-x^2}}$ 8) $\int_0^3 \frac{dx}{(x+3)^{\frac{5}{2}}}$

4. Aniq integral yordamida quyidagi chiziqlar bilan chegaralangan yuzani toping.

1) $x=y^2,\ y=x^2$ 2) $y=x^2,\ x+y=4$

3) $y=x^2,\ -x+y=4$ 4) $y=-x^2,\ y=x-4$

5) $y=-x^2,\ y=-x-4$ 6) $y=\sin x,\ y=\cos x,\ x=0$

7) $y=\sin x,\ y=\cos x,\ x=\frac{\pi}{6}$ 8) $y=-(x-2)^2-3,\ y=-x-8$

9) $y=2x^2,\ y=5$ 10) $\frac{x^2}{16}+\frac{y^2}{25}=1$

11) $y=arctg\sqrt{x},\ y+x^2=0,\ x=1$ 12) $y=e^x,\ x=0,\ y=e$

5. Quyidagi parametrik ko'rinishda berilgan chiziqlar bilan chegaralangan shaklning yuzini toping.

1) $x=12\cos t+5\sin t,\ y=5\cos t-12\sin t$

2) $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$ (sikloida) va $y=0$

3) $x=a(t-\sin t),\ y=a(1-\cos t)$ sikloidaning bir davri (orqasi) va Ox o'q.

4) $x=a\cos^3 t,\ y=a\sin^3 t$ (astroida)

5) $r^2=a^2\cos 2\varphi$ (lemniskata)

6) $x=a(1-\cos\varphi)$ (kardioida)

7) $x=\cos^4 t,\ y=\sin^4 t,\ \left(0\le t\le\frac{\pi}{2}\right)$

8) $x=\sin^4 t,\ y=\cos^2 t,\ \left(0\le t\le\frac{\pi}{2}\right)$

9) $r=\frac{\rho}{1+\cos\varphi},\ \left(|\varphi|\le\frac{\pi}{2}\right)$

10) $\varphi=\sqrt{r},\ (0\le r\le 5)$

11) $r=2(1+\cos\varphi),\ (r\le 1)$

12) $r = a(1-\sin\varphi)$, $\left(-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}\right)$

6. Quyidagi egri chiziqlarning yoylarining uzunliklarini aniqlang.

1) $y^2 = x^3$ egri chiziqning $x = \frac{4}{3}$ to'g'ri chiziq bilan kesilgan qismining uzunligi.

2) $x^2 + y^2 = 2$ egri chiziqing butun uzunligi.

3) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ egri chiziqning butun uzunligi.

4) $y^2 = (x+1)^3$ egri chiziqning $x = 4$ to'g'ri chiziq bilan kesilgan qismining uzunligi.

5) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ sikloida bir davrining uzunligi.

6) $y = \ln x$ ning $x = \frac{3}{4}$ dan $x = \frac{12}{5}$ gacha bo'lgan qismining uzunligi.

7) $y = e^x$ ning $x = 2$ dan $x = e$ gacha bo'lgan qismining uzunligi.

8) $r = a\varphi$ spiral birinchi gajagi yoyining uzunligi.

9) $r = \frac{\rho}{1+\cos\varphi}$, $\left(|\varphi| \le \frac{\pi}{2}\right)$

10) $\varphi = \frac{1}{2}\left(r + \frac{1}{r}\right)$, $(1 \le r \le 3)$

11) $\varphi = \sqrt{r}$, $(0 \le r \le 5)$.

12) $r = a\varphi$ (Arximed spirali), $0 \le \varphi \le 2\pi$.

13) $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \ln\pi$

14) $r = a\cos^3\frac{\varphi}{2}$, $0 \le \varphi \le \frac{\pi}{2}$

7. Aylanish sirtlarining yuzalarini hisoblash. Quyidagi chiziqlarni Ox o'qi atrofida aylantirishdan hosil bo'lgan aylanish sirtlarining yuzalarini hisoblang.

1) $y = \frac{x^3}{3}$ egri chiziqning $x = -3$ dan $x = 2$ gacha bo'lgan yoyi.

2) $y = 4 + x$ egri chiziqning $x = 2$ to'g'ri chiziq bilan kesishgan qismi

3) $x = \frac{t^3}{3}$, $y = 4 - \frac{t^2}{2}$ egri chiziqning koordinata o'qlari bilan kesishgan nuqtalari orasidagi qismi.

4) $x^2 + y^2 = 25$

5) $y = \sin x$ egri chiziqning bitta yarim to'lqini.

6) $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$ sikloidaning bir davri.

8. Quyidagi chiziqlarni Oy o'qi atrofida aylantirishdan hosil bo'lgan aylanish sirtining yuzalarini hisoblang.

1) $y=\frac{1}{2}x^2$ ning $y=1{,}5$ to'g'ri chiziq bilan kesishgan qismi.

2) $4x^2+y^2=4$ (ko'rsatma y ni erkli o'zgaruvchi deb olinsa, izlangan sirt yuzi, $S=\pi\int_0^2\sqrt{16-3y^2}\,dy$ bo'ladi. So'ngra $y=\frac{4}{\sqrt{3}}\sin t$ o'rniga qo'yishni tadbiq qilinadi).

3) $x=\ln\left(\sqrt{y}-\sqrt{y^2-1}\right)$, $\left(\frac{5}{4}\le y\le\frac{5}{3}\right)$

4) $4x+2\ln y=y^2$, $\left(e^{-1}\le y\le e\right)$

9. Quyidagi chiziqlar bilan chegaralangan figuralarning aylanishidan hosil bo'lgan jismlarning hajmlari aniqlansin.

1) $y^2=2px$ va $x=h$, Ox o'q atrofida.

2) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ va $y=\pm b$, Oy o'q atrofida.

3) $xy=4$, $x=1$, $x=4$, $y=0$, Ox o'q atrofida.

4) $y^2=(x+4)^3$ va $x=0$, Oy o'q atrofida.

5) $y^2=4-x$, $y=0$, Oy o'q atrofida.

6) $(y-a)^2=ax$, $x=0$, $y=2a$, Ox o'q atrofida.

7) $y=\cos x$ va $y=-1$, $-\pi\le x\le\pi$ bo'lganda $y=-1$ to'g'ri chiziq atrofida.

8) $y=x\sqrt{-x}$, $x=-4$ va $y=0$, Oy o'q atrofida.

9) $y=\sin x$ (bitta yarim to'lqinini), $y=0$, Ox o'q atrofida.

10) $x^2-y^2=4$, $y=\pm2$, Oy o'q atrofida.

11) $y=x^2$, $y=4$, $x=2$ to'g'ri chiziq atrofida.

12) $x=a(t-\sin t)$, $y=a(1-\cos t)$ sikloidaning bir davri, Ox o'q atrofida

10. Fizika masalalari.

1) $x=0$, $x=a$, $y=0$ va $y=b$ chiziqlar bilan chegaralangan to'g'ri to'rtburchakning Ox va Oy o'qlariga nisbatan inersiya momentlari aniqlansin.

2) $x=2$, $y=x^2$ va $y=0$ chiziqlar bilan chegaralangan yuzning Oy o'qqa nisbatan inersiya momenti topilsin.

3) $x=0$ va $x+y=a$ chiziqlar bilan chegaralangan uchburchakning Ox va Oy o'qlariga nisbatan statik momenti va og'irlik markazining koordinatalari topilsin.

(Ko'rsatma. Statik momentlar quyidagidan iborat $M_x=\int_0^a xydy$, $M_y=\int_a^a xydx$.

Og'rlik markazining koordinatalari: $x_c=\frac{M_y}{S}$, $y_c=\frac{M_x}{S}$, bunda S shaklning yuzi).

4) Uzunlig 1 m, kesim radiusi 2 mm bo'lgan mis sig'imni 0,001 m cho'zish uchun sarf etilgan ish hisoblansin.

5) $y=4-x^2$ va $y=0$ chiziqlar bilan chegaralangan yuzning og'irlik markazining koordinatalari topilsin.

6) Balandligi $H=2$m, asosining radiusi $R=0{,}3$ m ga teng konus shaklidagi chuqurdan (konusning uchi pastga qaragan) barcha suvni tortib chiqarish uchun bajarish kerak bo'lgan ish hisoblansin.

7) Agar 5 kg kuch prujinani 25 sm ga cho'zsa, u holda prujinani 60 sm ga cho'zish uchun qancha ish bajarish kerak.

8) $x=0$, $x=\frac{\pi}{2}$, $y=0$, $y=\cos x$ chiziqlar bilan chegaralangan figuraning og'irlik markazini toping.

10-§. SONLI QATORLAR

Ushbu

$$a_1, a_2, a_3, ..., a_n, ...$$

haqiqiy sonlar ketma-ketligi berilgan bo'lsin.

Ta'rif. Quyidagi

$$a_1+a_2+a_3+...+a_n+... \qquad (1)$$

ifoda ***qator*** (***sonli qator***) deyiladi.

(1) qator qisqacha $\sum_{n=1}^{\infty} a_n$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+...+a_n+....$$

Bunda $a_1, a_2, a_3, ..., a_n, ...$ elementlar ***qatorning hadlari*** deyiladi. a_n esa qatorning ***umumiy hadi*** deyiladi. (1) qatorning hadlaridan quyidagi

$$A_1=a_1,$$

$$A_2=a_1+a_2,$$

$$A_3=a_1+a_2+a_3,$$

$$\dots\dots\dots\dots\dots\dots\dots$$
$$A_n = a_1 + a_2 + a_3 + \dots + a_n ,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

yig‘indilarni tuzamiz. Bu yig‘indilar ***qatorning qismiy yig‘indilari*** deyiladi.

(1) qator berilgan holda har doim bu qatorning qismiy yig‘indilaridan iborat ushbu

$$\{A_n\}: A_1, A_2, A_3, \dots, A_n, \dots$$

sonlar ketma-ketligini hosil qilish mumkin.

Ta'rif. Agar $n \to \infty$ da (1) qatorning qismiy yig‘indilaridan iborat $\{A_n\}$ ketma-ketlik chekli limitga ega, ya'ni

$$\lim_{n\to\infty} A_n = A$$

bo‘lsa, u holda qator ***yaqinlashuvchi*** deyiladi.

Ta'rif. Agar $n \to \infty$ da (1) qatorning qismiy yig‘indilaridan iborat $\{A_n\}$ ketma-ketlik cheksiz bo‘lsa yoki bu limit mavjud bo‘lmasa, u holda (1) qator ***uzoqlashuvchi*** deyiladi.

(1) qatorning birinchi m ta hadini tashlasak, unda

$$a_{m+1} + a_{m+2} + \dots = \sum_{n=m+1}^{\infty} a_n$$

qator hosil bo‘ladi. Bu qator (1) qatorning (m -hadidan keyingi) ***qoldig‘i*** deyiladi.

10.1. Yaqinlashuvchi qatorlar haqidagi teoremalar

Biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \qquad (1)$$

qator berilgan bo‘lin.

Teorema. Agar (1) qator yaqinlashuvchi bo‘lsa, uning istalgan

$$a_{m+1} + a_{m+2} + \dots = \sum_{n=m+1}^{\infty} a_n$$

qoldig‘i ham yaqinlashuvchi bo‘ladi va aksincha.

$$a_{m+1} + a_{m+2} + \dots = \sum_{n=m+1}^{\infty} a_n$$

qoldiq qator yaqinlashuvchi bo‘lsa, berilgan (1) qator ham yaqinlashuvchi bo‘ladi.

Natija. Agar (1) qator yaqinlashuvchi bo‘lsa, uning qoldig‘i

$$r_m = a_{m+1} + a_{m+2} + \dots + a_{m+k} + \dots$$

$m \to \infty$ da nolga intiladi.

Teorema. Agar (1) qator yaqinlashuvchi bo‘lib, uning yig‘indisi A ga teng bo‘lsa, u holda

$$\sum_{n=1}^{\infty} ca_n = ca_1 + ca_2 + ca_3 + \ldots + ca_n + \ldots$$

qator ham yaqinlashuvchi va uning yig‘indisi cA ga teng bo‘ladi ($c \neq 0$- n ga bog‘liq bo‘lmagan o‘zgarmas son).

Teorema. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots,$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots + b_n + \ldots$$

qatorlar yaqinlashuvchi bo‘lib, ularning yig‘indilari mos ravishda A va B ga teng bo‘lsa, u holda

$$\sum_{n=1}^{\infty} (a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n) + \ldots$$

qator ham yaqinlashuvchi va uning yig‘indisi $A + B$ gat eng bo‘ladi.

Teorema (*qator yaqinlashishining zaruriy sharti*). Agar (1) qator yaqinlashuvchi bo‘lsa, bu qatorning a_n umumiy hadi $n \to \infty$ da nolga intiladi.

10.2. Sonli qatorlarning turlari

Qatorlarning tuzilishiga ko‘ra turlari quyidagilar:

1) barcha hadlarining ishoralari manfiy bo‘lmagan qatorlar;

2) biror hadidan boshlab, keyingi barcha hadlarining ishoralari manfiy bo‘lmagan qatorlar

3) barcha hadlarining ishoralari manfiy son yoki biror hadidan boshlab keyingi barcha hadlarining ishoralari manfiy bo‘lgan qatorlar;

4) cheksiz ko‘p manfiy ishorali va cheksiz ko‘p musbat ishorali hadlari bo‘lgan qatorlar.

10.3. Musbat qatorlarning yaqinlashuvchi bo‘lish sharti

Biror (1) qator berilgan bo‘lsin

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

Agar $a_n \geq 0$ $(n = 1, 2, 3, \ldots)$ bo‘lsa, u holda (1) qator ***musbat hadli qator*** yoki qisqacha, ***musbat qator*** deyiladi.

Teorema. Ushbu $\sum_{n=1}^{\infty} a_n$ musbat qator yaqinlashuvchi bo‘lishi uchun uning qismiy yig‘indilari ketma-ketligi yuqoridan chegaralangan bo‘lishi zarur va yetarli.

Misol. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ qator yaqinlashuvchidir.

Natija. Musbat qatorning qismiy yig‘indilaridan iborat ketma-ketlik yuqoridan chegaralanmagan bo‘lsa, qator uzoqlashuvchi bo‘ladi.

10.4. Musbat qatorlarni taqqoslash

Ikkita musbat $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qator berilgan bo‘lsin.

Teorema. n ning biror n_0 qiymatidan boshlab barcha $n \geq n_0$ lar uchun $a_n \leq b_n$ tengsizlik o‘rnli bo‘lsin. Agar *a*) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo‘ladi; *b*) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo‘ladi.

Teorema. Ushbu

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k \qquad (0 \leq k \leq \infty)$$

limit mavjud bo‘lsin. Agar *a*) $k < \infty$ va $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo‘ladi; *b*) $k > 0$ va $\sum_{n=1}^{\infty} b_n$ qator uzoqlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty} a_n$ qator ham uzoqlashuvchi bo‘ladi.

Natija. Agar ushbu

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k$$

limit o‘rinli bo‘lib, $0 < k < \infty$ bo‘lsa, u holda $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo‘ladi.

Teorema. $n \in N$ ning biror n_0 qiymatidan boshlab barcha $n > n_0$ lar uchun

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

tengsizlik o‘rinli bo‘lsin. U holda, agar *a*) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo‘ladi; *b*) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo‘ladi.

10.5. Musbat qatorlarning yaqinlashuvchilik alomatlari

Koshi alomati. Musbat qator $\sum_{n=1}^{\infty} a_n$ berilgan bo‘lsin. Agar $n \in N$ ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ qiymatlari uchun

$$\sqrt[n]{a_n} \leq q < 1 \quad (\sqrt[n]{a_n} \geq 1)$$

tengsizlik o‘rinli bo‘lsa, $\sum_{n=1}^{\infty} a_n$ qator yainlashuvchi (uzoqlashuvchi) bo‘ladi.

Agar ushbu

$$\lim_{n\to\infty} \sqrt[n]{a_n} = k$$

limit mavjud bo‘lsa, u holda $\sum_{n=1}^{\infty} a_n$ qator $k < 1$ bo‘lganda yaqinlashuvchi, $k > 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

Eslatma. Agar $\sum_{n=1}^{\infty} a_n$ qator uchun

$$\lim_{n\to\infty} \sqrt[n]{a_n} = k = 1$$

limit o‘rinli bo‘lsa, qator yaqinlashuvchi ham uzoqlashuvchi ham bo‘lishi mumkin.

Dalamber alomati. Agar $n \in N$ ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ qiymatlari uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1 \qquad \left(\frac{a_{n+1}}{a_n} \geq 1\right)$$

tengsizlik o‘rinli bo‘lsa, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi (uzoqlashuvchi) bo‘ladi.

Agar ushbu

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = d$$

limit mavjud bo‘lsa, u holda $d < 1$ bo‘lganda qator yaqinlashuvchi, $d > 1$ bo‘lganda esa qator uzoqlashuvchi bo‘ladi.

Raabe alomati. Agar $n \in N$ ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n > n_0$ qiymatlari uchun

$$n\left(1-\frac{a_{n+1}}{a_n}\right) \geq r > 1 \qquad \left(n\left(1-\frac{a_{n+1}}{a_n}\right) \leq 1\right)$$

tengsizlik o'rinli bo'lsa, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi (uzoqlashuvchi) bo'ladi.

Agar

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = R \quad \left(\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = R\right)$$

limit o'rinli bo'lsa, $R > 1$ bo'lsa qator yaqinlashuvchi, $R < 1$ bo'lsa qator uzoqlashuvchi bo'ladi.

Integral alomat (Koshining integral alomati). Ushbu $\sum_{n=1}^{\infty} a_n$ musbat qator berilgan bo'lsin. Faraz qilaylik, $[1, +\infty)$ oraliqda aniqlangan, uzluksiz o'smaydigan hamda manfiy bo'lmagan $f(x)$ funksiya uchun $F(x) = \int_1^x f(t)dt$ boshlang'ich funksiya va $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(x)$ bo'lsa, u holda

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_1^x f(t)dt$$

limit mavjud va chekli bo'lsa (1) qator yaqinlashuvchi, limit mavjud bo'lmasa yoki cheksiz bo'lsa (1) qator uzoqlashuvchi bo'ladi.

Gauss alomati. $\sum_{n=1}^{\infty} a_n$ musbat hadli qator uchun

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + \frac{\theta_n}{n^{1+\varepsilon}} \quad (|\theta_n| < \varepsilon, \varepsilon > 0)$$

bo'lsa, u holda

a) $\lambda > 1$ bo'lsa qator yaqinlashuvchi;

b) $\lambda < 1$ bo'lsa qator uzoqlashuvchi;

d) $\lambda = 1$ bo'lib, $\mu > 1$ bo'lsa qator yaqinlashuvchi;

e) $\lambda = 1$ bo'lib, $\mu \leq 1$ bo'lsa qator uzoqlashuvchi bo'ladi.

10.6. Garmonik qator

Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

qator garmonik qator deyiladi va u uzoqlashuvchi.

Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}=1+\frac{1}{2^{\alpha}}+\frac{1}{3^{\alpha}}+\ldots+\frac{1}{n^{\alpha}}+\ldots$$

qator ***umumlashgan garmonik qator*** deyiladi $\alpha>1$ da qator yaqinlashuvchi va $\alpha\le 1$ da uzoqlashuvchi.

10.7. Ixtiyoriy hadli qatorning yaqinlashuvchiligi

Biror ixtiyoriy $\sum_{n=1}^{\infty}a_n$ qator berilgan bo‘lsin.

Teorema. Ixtiyoriy $\sum_{n=1}^{\infty}a_n$ qator yaqinlashuvchi bo‘lishi uchun ixtiyoriy $\varepsilon>0$ son olinganda ham shunday $n_0\in N$ son mavjud bo‘lib, barcha $n>n_0$ va $m=1,2,3,\ldots$ lar uchun

$$|a_{n+1}+a_{n+2}+\ldots+a_{n+m}|<\varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

10.8. Qatorning absolyut va shartli yaqinlashuvchiligi

Ixtiyoriy $\sum_{n=1}^{\infty}a_n$ qator berilgan bo‘lsin. Bu qator hadlarining absolyut qiymatlaridan quyidagi

$$\sum_{n=1}^{\infty}|a_n|=|a_1|+|a_2|+|a_3|+\ldots+|a_n|+\ldots$$

qatorni tuzamiz.

Ta’rif. Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}a_n$ qator ***absolyut yaqinlashuvchi*** deyiladi.

Ta’rif. Agar $\sum_{n=1}^{\infty}a_n$ yaqinlashuvchi bo‘lib, $\sum_{n=1}^{\infty}|a_n|$ qator uzoqlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}a_n$ qator ***shartli yaqinlashuvchi*** deyiladi.

Teorema. Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}a_n$ qator ham yaqinlashuvchi bo‘ladi.

Eslatma. $\sum_{n=1}^{\infty}|a_n|$ qatorning uzoqlashuvchi bo‘lishidan $\sum_{n=1}^{\infty}a_n$ qatorning uzoqlashuvchi bo‘lishi har doim kelib chiqavermaydi.

Dalamber alomati. $\sum_{n=1}^{\infty} a_n$ qator uchun

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=D$$

bo‘lsin. U holda $\sum_{n=1}^{\infty} a_n$ qator $D<1$ bo‘lganda absolyut yaqinlashuvchi bo‘ladi.

10.9. Hadlarining ishoralari navbat bilan o‘zgarib keladigan qatorlar. Leybnis teoremasi

Ushbu

$$c_1-c_2+c_3-c_4+\ldots+(-1)^{n-1}c_n+\ldots$$

qatorni qaraylik, bunda $c_n>0$ $(n=1,2,3,\ldots)$.
Odatda bunday qator ***hadlarining ishoralari navbat bilan o‘zgarib keladigan qator*** deyiladi.

Misol. Ushbu

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}\frac{1}{n}+\ldots,$$

qator hadlarining ishoralari navbat bilan o‘zgarib keladigan qatordir.

Teorema (*Leybnis teoremasi*). Agar $c_1-c_2+c_3-c_4+\ldots+(-1)^{n-1}c_n+\ldots$ qatorda

$$c_{n+1}<c_n \qquad (n=1,2,3,\ldots)$$

tengsizlik o‘rinli bo‘lib,

$$\lim_{n\to\infty}c_n=0$$

bo‘lsa, $c_1-c_2+c_3-c_4+\ldots+(-1)^{n-1}c_n+\ldots$ qator yaqinlashuvchi bo‘ladi.

10.10. Yaqinlashuvchi qatorning xossalari. Riman teoremasi

1. *Guruhlash xossasi*. Biror $\sum_{n=1}^{\infty} a_n$ qator berilgan bo‘lsin. Bu qator hadlarini guruhlab quyidagi qatorni tuzamiz:

$$(a_1+a_2+\ldots+a_{n_1})+(a_{n_1+1}+a_{n_1+2}+\ldots+a_{n_2})+\ldots, \qquad (1)$$

bunda $n_1, n_2, \ldots$ $(n_1<n_2<\ldots)$ lar natural sonlar ketma-ketligi bo‘lib, $k\to\infty$ da $n_k\to\infty$.

Agar $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo‘lib, uning yig‘indisi A songa teng bo‘lsa, u holda bu qatorning hadlarini guruhlashdan hosil bo‘lgan

(1) qator ham yaqinlashuvchi va uning yig‘indisi ham A songa teng bo‘ladi.

2. *O‘rin almashtirish xossasi*. Ixtiyoriy $\sum_{n=1}^{\infty} a_n$ qator berilgan bo‘lsin. Bu qator hadlarini o‘rinlarini almashtirib, quyidagi

$$\sum_{n=1}^{\infty} a'_n = a'_1 + a'_2 + \ldots + a'_n + \ldots . \qquad (2)$$

qatorni tuzamiz. Bu (2) qatorning har bir a'_n hadi $\sum_{n=1}^{\infty} a_n$ qatorning tayin bir a_n hadining aynan o‘zidir.

Agar $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi bo‘’lib, yig‘indisi A songa teng bo‘lsa, u holda bu qator hadlarining o‘rinlarini ixtiyoriy ravishda almashtirishdan hosil bo‘lgan (2) qator yaqinlashuvchi bo‘ladi va uning yig‘indisi ham A songa teng bo‘ladi.

Teorema (*Riman teoremasi*). Agar $\sum_{n=1}^{\infty} a_n$ qator shartli yaqinlashuvchi bo‘lsa, u holda har qanday A (chekli yoki cheksiz) olinganda ham berilgan qator hadlarining o‘rinlarini shunday almashtirish mumkinki, hosil bo‘lgan qatorning yig‘indisi xuddi shu A ga teng bo‘ladi.

M7. Sonli qatorlarga doir mashqllar

1.Quyidagi qatorlarning umumiy hadini toping.

1) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \cdots$

2) $\frac{2}{3} + \frac{4}{9} + \frac{6}{27} + \frac{18}{31} + \cdots$

3) $\frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} + \frac{1}{30} - \cdots$

4) $1 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{10}} + \cdots$

5) $2 + \frac{2^2}{1\cdot 2} + \frac{2^3}{1\cdot 2\cdot 3} + \frac{2^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$

6) $1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \cdots$

7) $\frac{5}{6} + \frac{13}{36} + \frac{35}{216} + \cdots$

2. Quyidagi qatorlarning n ta hadlar yig’indisi S_n topilsin.

1) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2\,n-1)(2n+1)} + \cdots$

2) $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \cdots + \frac{1}{(3\,n-2)(3\,n+1)} + \cdots$

3) $\frac{3}{1^2\cdot 2^2} + \frac{5}{2^2\cdot 3^2} + \cdots + \frac{24+1}{n^2(n+1)^2} + \cdots$

4) $\frac{5}{6}+\frac{13}{36}+\cdots+\frac{3^n+2^n}{6\,n}+\cdots$

5) $\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2\,n+1}{n^2(n+1)^2}+\cdots$

2. Quyidagi qatorlarning zaruriy shartiga ko'ra uzoqlashuvchilikka tekshiring.

1) $0{,}001+\sqrt{0{,}001}+\sqrt[3]{0{,}001}+\sqrt[4]{0{,}001}+\cdots$

2) $\frac{3}{4}+\frac{5}{6}+\cdots+\frac{2\,n+1}{2\,n+2}+\cdots$

3) $\sqrt{\frac{3}{2}}+\sqrt{\frac{4}{3}}+\cdots+\sqrt{\frac{n+2}{n+1}}+\cdots$

4) $\frac{1}{2\sqrt{2}}+\frac{1}{4\sqrt{4}}+\frac{1}{6\sqrt{6}}+\cdots+\frac{1}{2\,n\sqrt{2\,n}}+\cdots$

5) $\frac{2}{3}+\frac{4}{9}+\frac{6}{27}+\cdots+\frac{2\,n}{3^{\,n}}+\cdots$

3. Taqqoslash belgisi yordamida qator yaqinlashuvchiligini aniqlang.

1) $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\cdots+\frac{n}{n^2+1}+\cdots$

2) $\frac{3}{1\cdot 4}+\frac{5}{4\cdot 9}+\frac{7}{9\cdot 16}+\cdots+\frac{2n+1}{n^2(n+1)^2}+\cdots$

3) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n+1}}+\cdots$

4) $1+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 5}+\frac{1}{n\cdot 5^{n-1}}+\cdots$

5) $\frac{1}{\ln 2}+\frac{1}{\ln 3}+\cdots+\frac{1}{\ln(n+1)}+\cdots$

4. Dalamber alomatiga asosan quyidagi qatorlarni yaqinlashishga tekshiring.

1) $\frac{2}{3}+\frac{4}{9}+\frac{6}{27}+\frac{8}{81}+\cdots$

2) $1+\frac{2}{2!}+\frac{4}{3!}+\frac{8}{4!}+\cdots$

3) $1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\cdots$

4) $1+\frac{3}{2\cdot 3}+\frac{3^2}{2^2\cdot 5}+\frac{3^3}{2^3\cdot 7}+\cdots$

5) $\frac{1}{2}+\frac{3!}{2\cdot 4}+\frac{5!}{2\cdot 4\cdot 6}+\frac{7!}{2\cdot 4\cdot 6\cdot 8}+\cdots$

6) $\sum_{n=1}^{\infty}\frac{n^3}{4^n}$

7) $\sum_{n=1}^{\infty}\frac{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-1)}{1\cdot 6\cdot 11\cdot\ldots\cdot(5n-4)}$

8) $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{3^n n!}$

9) $\sum_{n=1}^{\infty}\frac{(2n+1)!}{(3n+4)3^n}$

10) $\sum_{n=1}^{\infty}\frac{2\cdot 5\cdot\cdot\ldots\cdot(3n+2)}{2^n(n+1)}$

5. Koshi integral alomati bilan quyidagi qatorlarni yaqinlashishga tekshiring.

1) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$

2) $1+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{7}}+\frac{1}{\sqrt{10}}+\cdots$

3) $\frac{1}{2^3}+\frac{2}{3^3}+\frac{3}{4^3}+\cdots$

4) $\frac{1}{1+1^2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\cdots$

5) $\frac{1}{1+1^2}+\frac{2}{1+2^2}+\frac{3}{1+3^2}+\cdots$

6) $\frac{1}{3^2-1}+\frac{1}{5^2-1}+\frac{1}{7^2-1}+\cdots$

7) $\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^n$

8) $\sum_{n=1}^{\infty}2^n\left(\frac{n}{n+1}\right)^{n^2}$

9) $\sum_{n=1}^{\infty}\left(\frac{2n}{n+2}\right)^n$

10) $\sum_{n=1}^{\infty}\left(\frac{2n-1}{2n+1}\right)^{n(n-1)}$

6. Ishorasi o’zgaruvchi qatorlarning yaqinlashuvchanligini tekshiring.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\sqrt[3]{n}}$

2) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n+1}}$

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+2}{n}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)^3}$

5) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[4]{n}}$

6) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)}$

7) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$

8) $\sum_{n=1}^{\infty}\frac{(-1)^n\sin\frac{\pi}{n}}{n}$

9) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ln n}{\sqrt{n}}$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt[n]{n^2+1}}$

11-§. R^m FAZO. R^m FAZODA KETMA-KETLIK VA UNING LIMITI.

Ko'p o'zgaruvchili funksiyalar, ularning limiti va uzluksizligi

11.1. R^m Evklid fazosi

m ta haqiqiy sonlar to'plami R ning o'zaro Dekart ko'paytmasidan iborat ushbu

$$R^m = R\times R\times\ldots\times R = \{(x_1, x_2, \ldots, x_m); x_1 \in R, \ldots, x_m \in R\}$$

to'plam R^m fazo (m o'lchamli ***Evklid fazosi***) deb ataladi.

11.2. R^m fazoda masofa va uning limiti.

R^m to'plamda ixtiyoriy $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$ nuqtalarni olaylik. Ushbu

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_m - y_m)^2} = \sqrt{\sum_{k=1}^{m}(x_k - y_k)^2}$$

miqdorga x va y nuqtalar orasidagi masofa deyiladi.

U quyidagi xossalarga ega:

1^0. $\rho(x, y) \geq 0$ va $\rho(x, y) = 0 \Leftrightarrow x = y$,

2^0 $\rho(x, y) = \rho(y, x)$,

3^0 $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ $(z \in R^m)$.

11.3. R^m fazoda ketma-ketlik

Ushbu

$$f : N \to R^m$$

akslantirishning tasvirlari (obrazlari) dan tuzilgan

$x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots, (x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}), n \in N)$

to'plam R^m fazoda ***ketma-ketlik*** deyiladi va u $\{x^{(n)}\}$ kabi belgilanadi.

11.4. R^m fazoda ketma-ketlikning limiti

R^m fazoda biror $\{x^{(n)}\}: x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots$ ketma-ketlik va $a = (a_1, a_2, \ldots, a_m)$ nuqta berilgan bo'lsin.

Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday $n_0 \in N$ topilsaki, ixtiyoriy $n > n_0$ uchun

$\rho(x^{(n)}, a) < \varepsilon$

tengsizlik bajarilsa, a nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deyiladi va

$$\lim_{n\to\infty} x^{(n)} = a \text{ yoki } n \to \infty \text{ da } x^{(n)} \to a$$

kabi belgilanadi.

11.5. Ketma-ketlikning yaqinlashuvchiligi

Ta'rif. Agar $\{x^{(n)}\}$ ketma-ketlik limitga ega bo'lsa, u yaqinlashuvchi ketma- ketlik deyiladi.

Misol. R^m fazoda ushbu

$$\{x^n\} = \left\{\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right\}$$

ketma-ketlikning limiti a=(0,0,…,0) ekanini ko'rsating.

Yechish. Ixtiyoriy $\varepsilon > 0$ sonni olaylik. Shu ε ga ko'ra $n_0 = \left[\frac{\sqrt{m}}{\varepsilon}\right] + 1$ ni topamiz. Unda ixtiyoriy $n > n_0$ uchun

$$\rho(x^{(n)}, a) = \rho((\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}), (0,0,\dots,0)) = \sqrt{(\frac{1}{n} - 0)^2 + (\frac{1}{n} - 0)^2 + \dots + (\frac{1}{n} - 0)^2} =$$

$$= \sqrt{\frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2}} = \sqrt{\frac{m}{n^2}} = \frac{\sqrt{m}}{n} < \frac{\sqrt{m}}{n_0} = \frac{\sqrt{m}}{\left[\frac{\sqrt{m}}{\varepsilon}\right] + 1} < \varepsilon$$

bo'ladi. Demak, $\rho(x^{(n)}, a) < \varepsilon$.

Ta'rifga ko'ra

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty}(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) = (0,0,\dots,0) = a$$

bo'ladi.

Teorema. R^m fazoda $\{x^{(n)}\} = \{x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}\}$ ketma-ketlik $a = (a_1, a_2, \dots, a_m)$ ga intilishi:

$$\lim_{n\to\infty} x^{(n)} = a$$

uchun bir yo'la

$$\lim_{n\to\infty} x_1^{(n)} = a_1$$

$$\lim_{n\to\infty} x_2^{(n)} = a_2$$

$$\dots\dots\dots\dots\dots\dots$$

$$\lim_{n\to\infty} x_m^{(n)} = a_m$$

bo'lishi zarur va yetarlidir. Demak,

$$\lim_{n\to\infty} x^{(n)} = a \Leftrightarrow \begin{cases} \lim\limits_{n\to\infty} x_1^{(n)} = a_1 \\ \lim\limits_{n\to\infty} x_2^{(n)} = a_2 \\ \dots\dots\dots\dots \\ \lim\limits_{n\to\infty} x_m^{(n)} = a_m \end{cases}$$

11.6. Ko'p o'zgaruvchili funksiya tushunchasi

R^m fazoda biror M to'plamni qariylik: $M \subset R^m$.

Ta'rif. Agar M to'plamdagi har bir $x=(x_1, x_2,..., x_m)$ nuqtaga biror qoida yoki qonunga ko'ra bitta haqiqiy son y $(y \in R)$ mos qo'yilgan bo'lsa, M to'plamda ko'p o'zgaruvchili (m ta o'zgaruvchili) funksiya berilgan deyiladi va u

$$f:(x_1, x_2,..., x_m) \to y \quad \text{yoki} \quad y = f(x_1, x_2,..., x_m)$$

kabi belgilanadi. Bunda M–funksiyaning aniqlanish to'plami, $x_1, x_2,..., x_m$- funksiya argumentlari, y esa $x_1, x_2,..., x_m$ o'zgaruvchilarning funksiyasi deyiladi.

Masalan, $f - R^m$ –fazodagi har bir $x=(x_1, x_2,..., x_m)$ nuqtaga shu nuqta koordinatalari kvadratlarining yig'indisini mos qo'yuvchi qoida, ya'ni

$$f:\ x=(x_1, x_2,..., x_m) \to x_1^2 + x_2^2 + ... + x_m^2$$

bo'lsin. Bu holda $y = x_1^2 + x_2^2 + ... + x_m^2$ funksiyaga ega bo'lamiz. Bu funksiyaning aniqlanish to'plami $M = R^m$ bo'ladi.

Misol. Ushbu

$$z = \sqrt{(-1 - x^2 - y^2)(\sin^2 \pi x + \sin^2 \pi y)}$$

funksiyaning aniqlanish to'plamini toping. Bu funksiya x va y larning

$$\sin^2 \pi x + \sin^2 \pi y = 0 \qquad (\text{chunki } -1 - x^2 - y^2 < 0)$$

bo'ladigan qiymatlaridagina aniqlangan. Keyingi tenglikdan topamiz:

$$\begin{matrix} \sin^2 \pi x = 0 \Rightarrow x = p \\ \sin^2 \pi y = 0 \Rightarrow y = q \end{matrix} \qquad (p \in Z, q \in Z).$$

Berilgan funksiyaning aniqlanish to'plami

$$M=\{(p,q) \in R^2 : p \in Z, q \in Z\}$$

bo'ladi.

Ayrim muhim ta'riflarni keltiramiz.

Ta'rif. M to'plamning har bir nuqtasi uning ichki nuqtasi bo'lsa, bunday to'plam ***ochiq to'plam*** deyiladi.

Masalan, ochiq shar ochiq to'plam bo'ladi.

Ta'rif. x^0 nuqtaning istalgan sferik atrofi $U_\varepsilon(x^0)$ da M to'plamning x^0 dan farqli kamida bitta nuqtasi topilsaki, x^0 nuqta M to'plamning ***limit nuqtasi*** deyiladi.

Ta'rif. $M \in R^m$ to'plamning barcha limit nuqtalari shu to'plamga tegishli bo'lsa, M to'plam ***yopiq to'plam*** deyiladi.

Ta'rif. $U^0 = \{x \in R^m : \rho(x, o) < r\}$, ($o = (0, 0, ..., 0)$) topilsaki, $M \subset U^0$ bo'lsa, u holda M ***chegaralangan to'plam*** deyiladi.

11.7. Karrali limit

Ta'rif (***Geyne ta'rifi***). Agar M to'plamning nuqtalaridan tuzilgan, a ga intiluvchi har qanday $\{x^{(n)}\}$ ($x^{(n)} \neq a, n = 1, 2,...$) ketma-ketlik olinganda ham mos $\{f(x^{(n)})\}$ ketma-ketlik hamma vaqt yagona b songa (chekli yoki cheksiz) limitga intilsa, b son $f(x)$ funksiyaning a nuqtadagi limiti deb ataladi.

Ta'rif (***Koshi ta'rifi***). Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday δ >0 son topilsaki, ushbu $0 < \rho(x, a) < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $x \in M$ nuqtalarda

$$|f(x) - b| < \varepsilon$$

tengsizlik bajarilsa, b son $f(x)$ funksiyaning a nuqtadagi limiti deb ataladi.

Funksiya limiti

$$\lim_{x \to a} f(x) = b \quad \text{yoki} \quad \lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2 \\ \dots\dots \\ x_m \to a_m}} f(x_1, x_2, ..., x_m) = b$$

kabi belgilanadi.

Misol. Ushbu

$$f(x, y) = \begin{cases} x + y\sin\dfrac{1}{x}, & agar \quad x \neq 0 \quad bo'lsa, \\ 0, & agar \quad x = 0 \quad bo'lsa. \end{cases}$$

funksiyaning $(x, y) \to (0,0)$ dagi limiti nolga teng ekanini ko'rsating.

Ixtiyoriy $\varepsilon > 0$ songa ko'ra $\delta = \frac{\varepsilon}{2}$ deb olinsa, unda $0 < \rho((x, y),(0,0)) < \delta$ tengsizlikni qanoatlantiruvchi barcha (x, y) nuqtalarda

$$|f(x, y) - 0| = \left|x + y\sin\frac{1}{x}\right| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2} < 2\delta = \varepsilon$$

tengsizlik o‘rinli bo‘ladi. Bu esa Koshi ta’rifiga ko‘ra $(x,y)\to(0,0)$ da berilgan
funksiyaning limiti 0 ekanini bildiradi:

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}(x+y\sin\frac{1}{x})=0\,.$$

Ta’rif. Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta>0$ topilsaki, ushbu $\rho((x,y),(0,0))>\delta$ tengsizlikni qanoatlantiruvchi barcha (x,y) nuqtalarda

$$\left|f(x,y)-b\right|<\varepsilon$$

tengsizlik bajarilsa, b son $f(x,y)$ funksiyaning $x\to\infty, y\to\infty$ dagi limiti deyiladi va

$$\lim_{\substack{x\to\infty\\ y\to\infty}} f(x,y)=b$$

kabi belgilanadi.

Misol. Ushbu

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x^2+y^2)x^2y^2}{1-\cos(x^2+y^2)}$$

limitni hisoblang.

Avvalo

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x^2+y^2)x^2y^2}{1-\cos(x^2+y^2)}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x^2+y^2)x^2y^2}{2\sin^2\frac{(x^2+y^2)}{2}}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{\frac{x^2+y^2}{2}x^2y^2}{\sin^2\frac{x^2+y^2}{2}}=$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{\frac{x^2+y^2}{2}}{\sin\frac{x^2+y^2}{2}}\right]^2\cdot\frac{2x^2y^2}{x^2+y^2}=2\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{\frac{x^2+y^2}{2}}{\sin\frac{x^2+y^2}{2}}\right]^2\cdot\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^2}{x^2+y^2}\,.$$

ekanini topamiz. So‘ngra $x=r\cos\varphi, y=r\sin\varphi$ almashtirishni bajaramiz. Unda

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^2}{x^2+y^2}=\lim_{r\to 0}\frac{r^4\cos^2\varphi\sin^2\varphi}{r^2(\cos^2\varphi+\sin^2\varphi)}=\lim_{r\to 0}r^2\cos^2\varphi\sin^2\varphi=0$$

bo‘ladi. Demak,

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x^2+y^2)x^2y^2}{1-\cos(x^2+y^2)}=0\,.$$

11.8. Takroriy limit

Faraz qilaylik, $f(x_1, x_2,..., x_m)$ funksiya M to‘plamda $(M \subset R^m)$ berilgan bo‘lib, $a = (a_1, a_2,..., a_m)$ nuqta shu M to‘plamning limit nuqtasi bo‘lsin. $x_1, x_2,..., x_{i-1}, x_{i+1},..., x_m$ lar tayinlangan bo‘lib $x_i \to a_i$ da berilgan funksiyaning limiti (agar u mavjud bo‘lsa) $x_1, x_2,...x_m$ o‘zgaruvchilarga bog‘liq bo‘ladi:

$$\lim_{x_i \to a_i} f(x_1, x_2,...,x_m) = \varphi_i(x_1, x_2,...,x_{i-1}, x_{i+1},...,x_m).$$

φ_i funksiyalarda ham shunday mulohaza yuritib ushbu

$$\lim_{x_m \to a_m} \lim_{x_{m-1} \to a_{m-1}} \cdots \lim_{x_1 \to a_1} f(x_1, x_2,...,x_m)$$

ni hosil qilamiz. Odatda bu limit $f(x_1, x_2,..., x_m)$ funksiyaning ***takroriy limiti*** deyiladi.

Teorema. $f(x, y)$ funksiya $M = \{(x, y) \in R^2 : |x - x_0| < a_1, |y - y_0| < a_2\}$ to‘plamda berilgan bo‘lsin.

Agar: 1) $(x, y) \to (x_0, y_0)$ da $f(x, y)$ funksiyaning karrali limiti mavjud:

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = b;$$

2) har bir tayinlangan x da (har bir tayinlangan y da)

$$\lim_{y \to y_0} f(x, y) = \varphi(x) \qquad \left(\lim_{x \to x_0} f(x, y) = \phi(y)\right)$$

limit mavjud bo‘lsa, u holda

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y) = b \qquad \left(\lim_{y \to y_0} \lim_{x \to x_0} f(x, y) = b\right)$$

takroriy limit ham mavjud bo‘ladi.

Misol. Agar $f(x, y) = \dfrac{x^y}{1 + x^y}$ bo‘lsa, ushbu takroriy limitlarni hisoblang**.**

$$\lim_{x \to +\infty} \left(\lim_{y \to +0} f(x, y) \right) \text{ va } \lim_{y \to +0} \left(\lim_{x \to +\infty} f(x, y) \right)$$

Yechish: x ni o‘zgarmas desak $y > 0$ da x^y y-ning funksiyasi sifatida uzluksiz bo‘ladi, shu sababli

$$\lim_{y \to +0} x^y = 1$$

bo‘ladi.

y ning o‘zgarmas ($y > 0$) qiymatida, x ning barcha $x > 0$ qiymatida x^y x –ning funksiyasi sifatida uzluksizligidan

$$\lim_{y\to+\infty} x^y = +\infty$$

bo'ladi.

$$\lim_{x\to+\infty}\left(\lim_{y\to+0}\frac{x^y}{1+x^y}\right)=\lim_{x\to+\infty}\frac{1}{1+1}=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2},$$

$$\lim_{y\to+0}\left(\lim_{x\to+\infty}\frac{x^y}{1+x^y}\right)=\lim_{y\to+0}\left(\lim_{x\to+\infty}\frac{1}{\frac{1}{x^y}+1}\right)=\lim_{y\to+0}1=1.$$

12-§. KO'P O'ZGARUVCHILI FUNKSIYANING UZLUKSIZLIGI

12.1. Funksiya uzluksizligi ta'riflari

Ta'rif. Agar $x\to a$ da, ya'ni

$$x_1\to a_1$$
$$\dots\dots\dots$$
$$x_m\to a_m$$

da $f(x)=f(x_1,x_2,\dots,x_m)$ funksiyaning limiti mavjud bo'lib,

$$\lim_{x\to a} f(x)=f(a),$$

ya'ni

$$\lim_{\substack{x_1\to a_1\\ \dots\dots\\ x_m\to a_m}} f(x_1,x_2,\dots,x_m)=f(a_1,a_2,\dots,a_m)$$

bo'lsa, funksiya $a=(a_1,a_2,\dots,a_m)$ nuqtada uzluksiz deb ataladi.

Ta'rif (*Geyne ta'rifi*). Agar M to'plamning nuqtalaridan tuzilgan a ga ($a\in M$) intiluvchi har qanday $\{x^{(n)}\}$ ketma-ketlik olinganda ham mos $\{f(x^{(n)})\}$ ketma-ketlik hamma vaqt $f(a)$ ga intilsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

Ta'rif (*Koshi ta'rifi*). Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta>0$ topilsaki, $\rho(x,y)<\delta$ tengsizlikni qanoatlantiruvchi barcha $x\in M$ nuqtalarda

$$|f(x)-f(a)|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

$f(x)=f(x_1,x_2,\dots,x_m)$ funksiya argumentlarining orttirmalari

$$\Delta x_1,\Delta x_2,\dots\Delta x_m$$

ga mos ushbu

$$f(x)-f(a)=f(x_1,x_2,\ldots,x_m)-f(a_1,a_2,\ldots,a_m)=$$
$$=f(a_1+\Delta x_1,a_2+\Delta x_2,\ldots,a_m+\Delta x_m)-f(a_1,a_2,\ldots,a_m)$$

ayirma $f(x)$ funksiyaning a nuqtadagi to‘liq orttirmasi deyiladi va $\Delta f(a)$ kabi belgilanadi:

$$\Delta f(a)=f(a_1+\Delta x_1,a_2+\Delta x_2,\ldots,a_m+\Delta x_m)-f(a_1,a_2,\ldots,a_m).$$

Quyidagi

$$\Delta_{x_1}f(a)=f(a_1+\Delta x_1,a_2,\ldots a_m)-f(a_1,a_2,\ldots a_m),$$
$$\Delta_{x_2}f(a)=f(a_1,a_2+\Delta x_2,\ldots a_m)-f(a_1,a_2,\ldots a_m),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{x_m}f(a)=f(a_1,a_2,\ldots a_m+\Delta x_m)-f(a_1,a_2,\ldots a_m),$$

ayirmalar $f(x_1,x_2,\ldots,x_m)$ funksiyaning a nuqtadagi ***xususiy orttirmalari*** deyiladi.

Misol. Ushbu

$$f(x,y)=\frac{y}{x^2+y^2+5}$$

funksiyaning ixtiyoriy $(x_0,y_0)\in R^2$ nuqtada uzluksiz bo‘lishini ko‘rsating.

(x_0,y_0) nuqtaga $\Delta x,\Delta y$ orttirmalar berib, funksiyaning to‘liq orttirmasini topamiz:

$$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=\frac{y_0+\Delta y}{(x_0+\Delta x)^2+(y_0+\Delta y)^2+5}-\frac{y_0}{x_0^{\,2}+y_0^{\,2}+5}=$$
$$=\frac{(y_0+\Delta y)(x_0^{\,2}+y_0^{\,2}+5)-y_0\left[(x_0+\Delta x)^2+(y_0+\Delta y)^2+5\right]}{[(x_0+\Delta x)^2+(y_0+\Delta y)^2+5](x_0^{\,2}+y_0^{\,2}+5)}$$

bu tengliklardan

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta f(x_0,y_0)=0$$

bo‘lishi kelib chiqadi. Yuqoridagi ta’rifdan berilgan funksiya (x_0,y_0) nuqtada uzluksiz bo‘ladi.

12.2. Xususiy uzluksizlik

Ta’rif. Agar $\Delta x_k\to 0$ da funksiyaning xususiy orttirmasi $\Delta_{x_k}f$ ham nolga intilsa, ya’ni

$$\lim_{\Delta x_k\to 0}\Delta_{x_k}f=0 \qquad (k=1,2,\ldots,m)$$

bo‘lsa, $f(x_1,x_2,\ldots,x_m)$ funksiya $(x_1,x_2,\ldots,x_m)$ nuqtada x_k o‘zgaruvchisi bo‘yicha uzluksiz deyiladi. Odatda funksiyaning bunday uzluksizligini ***uning har o‘zgaruvchisi bo‘yicha xususiy uzluksizligi*** deyiladi.

Teorema. Agar $f(x_1, x_2, ..., x_m)$ funksiya $(x_1^0, x_2^0, ..., x_m^0) \in M$ nuqtada uzluksiz (barcha o‘zgaruvchili bo‘yicha bir yo‘la uzluksiz) bo‘lsa, funksiya shu nuqtada har bir o‘zgaruvchisi bo‘yicha xususiy uzluksiz bo‘ladi.

12.3. Funksiyaning uzilishi

Ta’rif. Agar

$$\lim_{\substack{x_1 \to a_1 \\ \cdots\cdots \\ x_m \to a_m}} f(x_1, x_2, ..., x_m) = b \neq f(a_1, a_2, ..., a_m)$$

bo‘lsa, yoki

$$\lim_{\substack{x_1 \to a_1 \\ \cdots\cdots \\ x_m \to a_m}} f(x_1, x_2, ..., x_m) = \infty$$

bo‘lsa, yoki $f(x_1, x_2, ..., x_m)$ funksiyaning limiti mavjud bo‘lmasa, u holda funksiya $(a_1, a_2, ..., a_m)$ nuqtada ***uzilishga*** ega deyiladi.

12.4. Funksiyaning tekis uzluksizligi

Ta’rif. Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, M to‘plamning $\rho((x_1', x_2', ..., x_m'), (x_1'', ..., x_m'')) < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $(x_1', ..., x_m') \in M, (x_1'', ..., x_m'') \in M$ nuqtalarda

$$\left| f(x_1'', x_2'', ..., x_m'') - f(x_1', x_2', ..., x_m') \right| < \varepsilon$$

tengsizlik bajarilsa, $f(x_1, x_2, ..., x_m)$ funksiya M to‘plamda ***tekis uzluksiz*** deyiladi.

Misol. Ushbu

$$f(x, y) = ax + by + c$$

funksiyaning

$$M = \{(x, y) \in R^2 : |x| < +\infty, |y| < +\infty, \quad a \in R, b \in R, c \in R\}$$

to‘plamda tekis uzluksiz bo‘lishini ko‘rsating.

Yechish: $(x_1, y_1) \in M$ va $(x_2, y_2) \in M$ nuqtalar uchun quyidagiga ega bo‘lamiz

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)| = |ax_1 + by_1 + c - (ax_2 + by_2 + c)| = \\ &= |a(x_1 - x_2) + b(y_1 - y_2)| \le |a| \cdot |x_1 - x_2| + |b| \cdot |y_1 - y_2| \end{aligned}.$$

Ixtiyoriy $\varepsilon > 0$ sonni olib, unga ko‘ra olinadigan $\delta > 0$ sonda

$$|x_1 - x_2| < \delta, \quad |y_1 - y_2| < \delta, \quad (\delta = \frac{\varepsilon}{2d}, d = \max(|a|, |b|))$$

shart bajarilganda

$$|f(x_1, x_2) - f(y_1, y_2)| \le d(|x_1 - x_2| + |y_1 - y_2|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

bo‘lib, ta’rifdan berilgan funksiya M da tekis uzluksizligi kelib chiqadi.

Teorema (*Kantor teoremasi*). Agar $f(x_1, x_2, ..., x_m)$ funksiya chegaralangan yopiq M to‘plamda ($M \subset R$) berilgan va uzluksiz bo‘lsa, funksiya shu to‘plamda tekis uzluksiz bo‘ladi.

M8. Ko’p o’zgaruvchili funksiyalarga doir mashqlar

1. Quyidagi funksiyalarning aniqlanish sohasini toping.

1) $z = \frac{x}{x+y}$

2) $z = \frac{y}{x}$

3) $z = \sqrt{4 - x^2 - y^2}$

4) $z = \sqrt{1 - \frac{x^2}{16} - \frac{y^2}{9}}$

5) $z = x\sqrt{x} + y\sqrt{y}$

6) $z = \sqrt{x} + \sqrt{y}$

7) $z = \frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{x-y}}$

8) $z = arcsin\frac{x^2+y^2}{9}$

9) $z = \sqrt{x - \sqrt{y}}$

10) $z = \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$

11) $z = \ln(\frac{x^2}{25} - \frac{y^2}{16} - 1)$

2. Ko’p o’zgaruvchili funksiyalarning limitini hisoblang.

1) $\lim\limits_{\substack{x\to0\\ y\to0}} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$

2) $\lim\limits_{\substack{x\to0\\ y\to0}} \frac{\sin(x^4y^2)}{(x^2+y^2)^2}$

3) $\lim\limits_{\substack{x\to0\\ y\to0}} \frac{(x^2+y^2)x^2y^2}{1-\cos(x^2+y^2)}$

4) $\lim\limits_{\substack{x\to0\\ y\to0}} \frac{e^{\frac{1}{x^4+y^4}}}{x^4+y^4}$

5) $\lim\limits_{\substack{x\to0\\ y\to0}} (1 + x^2y^2)^{\frac{1}{x^2+y^2}}$

6) $\lim\limits_{\substack{x\to0\\ y\to0}} \frac{(x+y)^2}{(x^2+y^2)^2}$

7) $\lim\limits_{\substack{x\to0\\ y\to0}} (x^2 + y^2) sin\frac{1}{x^2+y^2}$

8) $\lim\limits_{\substack{x\to 0\\ y\to 2}} \frac{\sin xy}{x}$

9) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x}{x+y}$

10) $\lim\limits_{\substack{x\to \infty\\ y\to k}} (1+\frac{y}{x})^x$

3. (0,0) nuqtada quyidagi funksiyalarning uzluksizligini tekshiring.

1) $f(x,y) = \frac{xy}{\sqrt{xy+1}-1}$, $f(0,0) = 2$

2) $f(x,y) = \frac{\sin(xy)}{x}$, $f(0,0) = 0$

3) $f(x,y) = \frac{x^2+xy+y^2}{x^2-y^2}$, $f(0,0) = 0$

4) $f(x,y) = \frac{1}{x^2+y^2}$, $f(0,0) = 0$

4. Quyidagi funksiyalarni uzilish nuqtalarini toping.

1) $z = \ln\sqrt{x^2+y^2}$

2) $z = \frac{1}{(x-y)^2}$

3) $z = \frac{1}{1-x^2-y^2}$

5.Takroriy limitlarni hisoblang.

1) Ushbu

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

funksiyaning (0,0) nuqtadagi takroriy limitlarini toping.

2) $f(x,y) = \frac{x^2+xy+y^2}{x^2-xy+y^2}$; $x_0 = 0, y_0 = 0.$

3) $f(x,y) = \frac{\sin(x+y)}{2x+3y}$; $x_0 = 0,\ y_0 = 0.$

4) $f(x,y) = \frac{cosx-cosy}{x^2+y^2}$; $x_0 = 0,\ y_0 = 0.$

5) $f(x,y) = \frac{sin3x-tg2y}{6x+3y}$; $x_0 = 0,\ y_0 = 0.$

6) $f(x,y) = \frac{x^2+y^2}{x^2+y^4}$; $x_0 = \infty,\ y_0 = \infty.$

7) $f(x,y) = \frac{x^y}{1+x^y}$; $x_0 = \infty,\ y_0 = 0.$

8) $f(x,y) = sin\frac{\pi x}{2x+3y}$; $x_0 = \infty,\ y_0 = \infty.$

9) $f(x,y) = \frac{sinx+cosy}{x+y}$; $x_0 = 0,\ y_0 = 0.$

10) $f(x,y) = \frac{1}{xy}tg\frac{xy}{1+xy}$; $x_0 = 0,\ y_0 = \infty.$

13-§. KO'P O'ZGARUVCHILI FUNKSIYANING HOSILASI VA DIFFERENSIALLARI

13.1. Ko'p o'zgaruvchili funksiyaning xususiy hosilalari

$f(x_1, x_{2,}..x_m)$ funksiya ochiq M to'plamda $(M \subset R^m)$ berilgan bo'lib, $(x_1^0, x_2^0, ...x_m^0) \in M$ bo'lsin. Bu funksiyaning x_k $(k = 1,2,...,m)$ koordinatasiga shunday Δx_k $(k = 1, 2,...,m)$ orttirma beraylikki, $(x_1^0, x_2^0,...,x_k^0 + \Delta x_k,...,x_m) \in M$ bo'lsin. Unda funksiya

$$\Delta x_k f = f(x_1^0, x_2^0,..,x_k^0 + \Delta x_k,..,x_m) - f(x_1^0, x_2^0,..,x_m^0)$$

xususiy orttirmaga ega bo'ladi.

Ta'rif. Agar $\Delta x_k \to 0$ da ushbu

$$\lim_{\Delta x_k \to 0} \frac{\Delta x_k f}{\Delta x_k} = \lim_{\Delta x_k \to 0} \frac{f(x_1^0,...,x_k^0 + \Delta x_k,..,x_m^0) - f(x_1^0,..,x_m^0)}{\Delta x_k}$$

limit mavlud va chekli bo'lsa, bu limit $f(x_1, x_2,.., x_m)$ funksiyaning $(x_1^0, x_2^0,...,x_m^0)$ nuqtadagi x_k o'zgaruvchisi bo'yicha ***xususiy hosilasi*** deyiladi va

$$\frac{\partial f(x_1^0,...,x_m^0)}{\partial x_k}, \frac{\partial f}{\partial x_k}, f`_{x_k}(x_1^0, x_2^0,...,x_m^0)$$

belgilarning biri bilan belgilanadi. Demak,

$$\frac{\partial f}{\partial x_k} = \lim_{\Delta x_k \to 0} \frac{\Delta x_k f}{\Delta x_k} \qquad (k = 1, 2,...,m).$$

Misol. Ushbu

$$f(x, y) = e^{x+y}$$

funksiyaning (2,2) nuqtada f'_x, f'_y xususiy hosilalarini hisoblang.

Ta'rifdan

$$\frac{\partial f(2,2)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(2 + \Delta x, 2) - f(2,2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{2+\Delta x+2} - e^{2+2}}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{e^{4+\Delta x} - e^4}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^4(e^{\Delta x} - 1)}{\Delta x} = e^4 \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^4$$

Xuddi shunga o'xshash,

$$\frac{\partial f(2,2)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(2, 2 + \Delta y) - f(2,2)}{\Delta y} = \lim_{\Delta y \to 0} \frac{e^{4+\Delta y} - e^4}{\Delta y} = e^4.$$

Demak,

$$\frac{\partial f(2,2)}{\partial x} = e^4, \frac{\partial f(2,2)}{\partial y} = e^4.$$

13.2. Ko‘p o‘zgaruvchili funksiyaning differensiali

Ta’rif. Agar $f(x_1, x_2, ..., x_m)$ funksiyaning $(x_1^0, x_2^0, ..., x_m^0)$ nuqtadagi $\Delta f(x_1^0, x_2^0, .. x_m^0)$ orttirmasini

$$\Delta f(x_1^0, x_2^0, ..., x_m^0) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m$$

kabi ifodalash mumkin bo‘lsa, $f(x_1, x_2, ..., x_m)$ funksiya $(x_1^0, x_2^0, ..., x_m^0)$ nuqtada differensiallanuvchi deyiladi (bunda $A_1, A_2, ..., A_m$ lar $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga bog‘liq bo‘lmagan o‘zgarmaslar, $\alpha_1, \alpha_2, ..., \alpha_m$ lar esa $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga bog‘liq va $\Delta x_1 \to 0, \Delta x_2 \to 0, ..., \Delta x_m \to 0$ da $\alpha_1 \to 0, \alpha_2 \to 0, ..., \alpha_m \to 0$ $(\Delta x_1 = \Delta x_2 = ... = \Delta x_m = 0)$ bo‘lganda $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$ deb olinadi).

Misol. Ushbu

$$f(x, y) = x^2 + y^2$$

funksiyaning ixtiyoriy $(x_0, y_0) \in R^2$ nuqtada differensiallanuvch ekanini ko‘rsating.

Berilgan funksiyaning (x_0, y_0) nuqtadagi to‘la orttirmasini topamiz:

$$\Delta f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = (x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 - (x_0^2 + y_0^2) =$$
$$= 2x_0\Delta x + 2y_0\Delta y + \Delta x^2 + \Delta y^2.$$

Agar $A_1 = 2x_0, A_2 = 2y_0, \alpha_1 = \Delta x, \alpha_2 = \Delta y$ deyilsa, unda

$$\Delta f(x_0, y_0) = A_1\Delta x + A_2\Delta y + \alpha_1\Delta x + \alpha_2\Delta y$$

bo‘ladi. Bu esa berilgan funksiyaning (x_0, y_0) nuqtada differensiallanuvchi ekanini bildiradi.

Ta’rif. $f(x_1, x_2, ..., x_m)$ funksiya orttirmasi $\Delta f(x_1^0, x_2^0, ..., x_m^0)$ ning $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli bosh qismi

$$A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + .. + \frac{\partial f}{\partial x_m}\Delta x_m$$

$f(x_1, x_2, ..., x_m)$ funksiyaning $(x_1^0, x_2^0, ..., x_m^0)$ nuqtadagi differensiali deyiladi va df yoki $df(x_1^0, x_2^0, ..., x_m^0)$ kabi belgilanadi.

Demak,

$$df(x_1^0, x_2^0, .., x_m^0) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m \quad (\Delta x_1 = dx_1, ..., \Delta x_m = dx_m).$$

13.3. Taqribiy hisoblashda to'liq differensialning tadbig'i

Faraz qilaylik, $f(x_1, x_2,..., x_m)$ funksiya ochiq $M \subset R^m$ to'plamda berilgan bo'lib, $(x_1^0, x_2^0,..,x_m^0) \in M$ nuqtada differensiallanuvchi bo'lsin. U holda

$$\Delta f\ (x_1^0, x_2^0,..,x_m^0) = df\ (x_1^0, x_2^0,..,x_m^0) + 0(\rho)$$

bo'ladi. $\rho \to 0$ da

$$\frac{\Delta f\left(x_1^0, x_2^0,...,x_m^0\right)}{df\left(x_1^0, x_2^0,...,x_m^0\right)} \to 1.$$

Natijada ushbu

$$\Delta f\ (x_1^0, x_2^0,..,x_m^0) \approx df\ (x_1^0, x_2^0,..,x_m^0)$$

taqribiy formulaga kelamiz. Uni

$$\Delta f\ (x_1^0, x_2^0,..,x_m^0) \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + ...\frac{\partial f}{\partial x_m}\Delta x_m$$

yoki

$$f\ (x_1^0 + \Delta x_1, x_2^0 + \Delta x_2,...x_m^0 + \Delta x_m) \approx f\ (x_1^0, x_2^0,..,x_m^0) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + ...\frac{\partial f}{\partial x_m}\Delta x_m$$

kabi yozish ham mumkin.

Misol. Ushbu

$$\alpha = 1{,}02^{3{,}01}$$

miqdorning taqribiy qiymatini toping. Berilgan miqdorning taqribiy qiymatini topish uchun

$$f(x, y) = x^y$$

funksiyani qaraymiz. Bu funksiya (1,3) nuqtada differensiallanuvchi. Demak,

$$\Delta f(1{,}3) = \frac{\partial f(1{,}3)}{\partial x}\Delta x + \frac{\partial f(1{,}3)}{\partial y}\partial y + 0(\rho).$$

Endi $\Delta x = 0{,}02, \Delta y = 0{,}01$ deylik: Unda

$$\Delta f(1{,}3) \approx \frac{\partial f(1{,}3)}{\partial x}\Delta x + \frac{\partial f(1{,}3)}{\partial y}\Delta y$$

bo'lib

$$f(1 + 0{,}02{,}3 + 0{,}01) - f(1{,}3) \approx y \cdot x^{y-1} \cdot \Delta x + x^y \ln x \cdot \Delta y\Big|_{x=1, y=3, \Delta x=0{,}02, \Delta y=0{,}01}$$

bo'ladi. Bundan

$$f(1{,}02;3{,}01) - f(1{,}3) \approx 3 \cdot 1 \cdot 0{,}02 + 1 \cdot \ln 1 \cdot 0{,}01 \Rightarrow 1{,}02^{3{,}01} - 1 \approx 0{,}06 \Rightarrow 1{,}02^{3{,}01} \approx 1{,}06.$$

Demak,

$$\alpha = 1{,}02^{3{,}01} \approx 1{,}06.$$

13.4. Yo'nalish bo'yicha hosila

Ta'rif. l chiziqdagi (x,y) nuqta l chiziq bo'ylab (x_0,y_0) nuqtaga intilganda ushbu

$$\frac{f(x,y)-f(x_0,y_0)}{\rho((x_0,y_0),(x,y))}$$

nisbatning limiti mavjud bo'lsa, bu limit $f(x,y)$ funksiyaning (x_0,y_0) nuqtadagi l ***yo'nalish bo'yicha hosilasi*** deyiladi va

$$\frac{\partial f(x_0,y_0)}{\partial l}$$

kabi belgilanadi. Demak,

$$\frac{\partial f}{\partial l}=\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)}{\rho((x_0,y_0),(x,y))}.$$

Teorema. Agar $f(x,y)$ funksiya (x_0,y_0) nuqtada differensiallanuvchi bo'lsa, u holda funksiya shu nuqtada har qanday l yo'nalish bo'yicha hosilaga ega va

$$\frac{\partial f(x_0,y_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0)}{\partial y}\cos\beta$$

bo'ladi.

Eslatma. Funksiyaning differensiallanuvchi bo'lmagan nuqtada ham yo'nalish bo'yicha hosila mavjud bo'lishi mumkin.

13.5. Murakkab funksiyaning hosilasi

Teorema. Agar

$$\begin{aligned} x_1&=\varphi_1(t_1,t_2,\ldots,t_k),\\ x_2&=\varphi_2(t_1,t_2,\ldots,t_k),\\ &\ldots\ldots\ldots\ldots\ldots\\ x_m&=\varphi_m(t_1,t_2,\ldots,t_k). \end{aligned}$$

funksiyalarning har biri $(t_1^0,t_2^0,\ldots,t_k^0)$ nuqtada differensiallanuvchi bo'lib, $f(x_1,x_2,\ldots,x_m)$ funksiya esa mos $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqtada differensiallanuvchi bo'lsa, u holda murakkab funksiya ham $(t_1^0,t_2^0,\ldots,t_k^0)$ nuqtada differensiallanuvchi bo'lib,

$$\begin{aligned} \frac{\partial f}{\partial t_1}&=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_1}+\ldots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_1},\\ \frac{\partial f}{\partial t_2}&=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_2}+\ldots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_2},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \frac{\partial f}{\partial t_k}&=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_k}+\ldots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_k}. \end{aligned}$$

bo‘ladi.

Misol. Ushbu

$$u = f(x, xy, xyz)$$

funksiyaning x, y, z argumentlar bo‘yicha hosilasini toping.

Yechish: Bu funksiya x, y, z o‘zgaruvchilarning murakkab funksiyasi: $u = f(x_1, x_2, x_3)$, bu yerda $x_1 = x,\ x_2 = xy,\ x_3 = xyz,\ u = f(x_1, x_2, x_3)$ funksiyaning x_1, x_2, x_3 argumentlar bo‘yicha hosilasi f_x, f_y, f_z bilan belgilaymiz.

Bu funksiyalar argumentlari ham xuddi f funksiyaning argumentlaridek yuqoridagi formulani qo‘llab quyidagiga ega bo‘lamiz.

$$\frac{\partial u}{\partial x} = f_1^{`} \cdot 1 + f_2^{`} \cdot y + f_3^{`} yz$$

$$\frac{\partial u}{\partial y} = f_2^{`} \cdot x + f_3^{`} xz$$

$$\frac{\partial u}{\partial z} = f_3^{`} xy.$$

13.6. Funksiyaning yuqori tartibli xususiy hosilalari

$f(x_1, x_2, ..., x_m)$ funksiya ochiq M $(M \subset R^m)$ to‘plamda berilgan bo‘lib, uning $(x_1, x_2, ..., x_m)$ nuqtasida $f'_{x_1}, f'_{x_2}, ..., f'_{x_m}$ xususiy hosilalarga ega bo‘lsin. Ma’lumki, bu xususiy hosilalar $x_1, x_2, ..., x_m$ larga bog‘liq bo‘ladi.

Ta’rif. $f'_{x_1}, f'_{x_2}, ..., f'_{x_m}$ larning x_k $(k = 1, 2, ..., m)$ o‘zgaruvchisi bo‘yicha xususiy hosilalari berilgan funksiyaning ***ikkinchi tartibli xususiy*** hosilalari deyiladi va

$$f''_{x_1 x_k}, ..., f''_{x_m x_k} \quad (k = 1, 2, ..., m)$$

yoki

$$\frac{\partial^2 f}{\partial x_1 \partial x_k}, \frac{\partial^2 f}{\partial x_2 \partial x_k}, ..., \frac{\partial^2 f}{\partial x_m \partial x_k} \quad (k = 1, 2, ..., m)$$

kabi belgilanadi. Demak,

$$\frac{\partial^2 f}{\partial x_1 \partial x_k} = f''_{x_1 x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_1}\right),$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_k} = f''_{x_2 x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_2}\right),$$

....

$$\frac{\partial^2 f}{\partial x_m \partial x_k} = f''_{x_m x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_m}\right).$$

Teorema. $f(x,y)$ funksiya ochiq M $(M \subset R^2)$ to‘plamda berilgan bo‘lib, shu to‘plamda $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ hamda $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ aralash hosilalarga ega bo‘lsin. Agar aralash hosilalar $(x_0, y_0) \in M$ nuqtada uzluksiz bo‘lsa, u holda shu nuqtada

$$\frac{\partial f^2(x_0,y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$$

bo‘ladi.

Misol. Ushbu $F = f(x+y, x^2+y^2)$ funksiyaning ikkinchi tartibli xususiy hosilasini toping. Quyidagi almashtirishni bajaramiz: $u = x+y, v = x^2+y^2$

$$\frac{\partial u}{\partial x} = 1; \frac{\partial^2 u}{\partial x^2} = 0; \frac{\partial u}{\partial y} = 1; \frac{\partial^2 u}{\partial y^2} = 0;$$

$$\frac{\partial v}{\partial x} = 2x; \frac{\partial^2 v}{\partial x^2} = 2; \frac{\partial v}{\partial y} = 2y; \frac{\partial^2 v}{\partial y^2} = 2;$$

$$\frac{\partial^2 u}{\partial x \partial y} = 0; \frac{\partial^2 v}{\partial x \partial y} = 0;$$

$$\frac{\partial^2 F}{\partial x^2} = \left(1 \cdot \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}\right)^2 F + 0\frac{\partial F}{\partial u} + 2\frac{\partial F}{\partial v} =$$

$$= \left(\frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}\right)^2 F + 2\frac{\partial F}{\partial v} = \frac{\partial^2 F}{\partial u^2} + 4x\frac{\partial^2 F}{\partial u \partial v} + 4x^2\frac{\partial^2 F}{\partial v^2} + 2\frac{\partial F}{\partial v};$$

$$\frac{\partial^2 F}{\partial y^2} = \left(1 \cdot \frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}\right)^2 F + 0\frac{\partial F}{\partial u} + 2\frac{\partial F}{\partial v} =$$

$$= \left(\frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}\right)^2 F + 2\frac{\partial F}{\partial v} = \frac{\partial^2 F}{\partial u^2} + 4y\frac{\partial^2 F}{\partial u \partial v} + 4y^2\frac{\partial^2 F}{\partial v^2} + 2\frac{\partial F}{\partial v};$$

$$\frac{\partial^2 F}{\partial x \partial y} = \left(1 \cdot \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}\right)\left(1 \cdot \frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}\right)F + 0\frac{\partial F}{\partial u} + 0\frac{\partial F}{\partial v} =$$

$$= \frac{\partial^2 F}{\partial u^2} + 2y\frac{\partial^2 F}{\partial u \partial v} + 2x\frac{\partial^2 F}{\partial v^2} + 4xy\frac{\partial^2 F}{\partial v^2} = \frac{\partial^2 F}{\partial u^2} + 2(x+y)\frac{\partial^2 F}{\partial u \partial v} + 4xy\frac{\partial^2 F}{\partial v^2}.$$

13.7. Funksiyaning yuqori tartibli differensiallari

Faraz qilaylik $f(x_1, x_2, ..., x_m)$ funksiya $(x_1, x_2, ..., x_m) \in R^m$ nuqtada ikki marta differensiallanuvchi bo‘lsin.

Ta’rif. $f(x_1, x_2, ..., x_m)$ funksiya $(x_1, x_2, ..., x_m) \in R^m$ nuqtada n marta differensiallanuvchi bo‘lganda, shu nuqtadagi $(n-1)$ tartibli differensiali $d^{n-1} f$ ning differensiali berilgan $f(x_1, x_2 ..., x_m)$ funksiyaning n -tartibli differensiali deyiladi va u $d^n f$ kabi belgilanadi. Demak,

$$d^n f = d(d^{n-1} f).$$

13.8. Ko‘p o‘zgaruvchili funksiyaning Teylor formulasi

$f(x_1, x_2,..., x_m)$ funksiya R^m fazoning $(x_1^0, x_2^0,...,x_m^0)$ nuqtasi atrofida $n+1$ marta differensiallanuvchi bo‘lsin. Ushbu formula

$$f(x_1,x_2,...,x_m) = f(x_1^0,x_2^0,..,x_m^0) + \frac{\partial f}{\partial x_1}(x_1 - x_1^0) + + \frac{\partial f}{\partial x_2}(x_2 - x_2^0) + ... +$$

$$+\frac{\partial f}{\partial x_m}(x_m - x_m^0) + \frac{1}{2!}\left(\frac{\partial}{\partial x_1}(x_1 - x_1^0) + \frac{\partial}{\partial x_2}(x_2 - x_2^0) + ... + \frac{\partial}{\partial x_m}(x_m - x_m^0)\right)^2 f +$$

$$+ ... + \frac{1}{n!}\left(\frac{\partial}{\partial x_1}(x_1 - x_1^0) + ... + \frac{\partial}{\partial x_m}(x_m - x_m^0)\right)^n f + R_n(f),$$

$$R_n(f) = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}(x_1 - x_1^0) + \frac{\partial}{\partial x_2}(x_2 - x_2^0) + ... + \frac{\partial}{\partial x_m}(x_m - x_m^0)\right)^{n+1} f$$

ko‘p o‘zgaruvchili funksiyaning ***Teylor formulasi***, $R_n(f)$ esa ***Teylor formulasining qoldiq hadi*** deyiladi.

Misol. Ushbu

$$f(x, y) = x^y$$

funksiyaning $n = 3$ bo‘lganda $(x_0, y_0) = (1,1)$ nuqta atrofida Teylor formulasini yozing.

Bu holda $f(x, y)$ funksiyaning Teylor formulasi quyidagicha bo‘ladi:

$$f(x,y) = f(x_0,y_0) + \left(\frac{\partial}{\partial x}(x - x_0) + \frac{\partial}{\partial y}(y - y_0)\right) f +$$

$$+\frac{1}{2!}\left(\frac{\partial}{\partial x}(x - x_0) + \frac{\partial}{\partial y}(y - y_0)\right)^2 f + \frac{1}{3!}\left(\frac{\partial}{\partial x}(x - x_0) + \frac{\partial}{\partial y}(y - y_0)\right)^3 f + R_3(f)$$

funksiyaning (1,1) dagi qiymati $f(1,1) = 1$.

Endi $f(x, y) = x^y$ funksiyaning xususiy hosilalarini va ularning (1,1) nuqtadagi qiymatlarini topamiz:

$$\frac{\partial f}{\partial x} = yx^{y-1}, \quad \frac{\partial f(1,1)}{\partial x} = 1,$$

$$\frac{\partial f}{\partial y} = x^y \ln x, \quad \frac{\partial f(1,1)}{\partial y} = 0,$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \qquad \frac{\partial^2 f(1,1)}{\partial x^2} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + yx^{y-1}\ln x, \qquad \frac{\partial^2 f(1,1)}{\partial x \partial y} = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = x^y \ln^2 x, \qquad \frac{\partial^2 f(1,1)}{\partial y^2} = 0,$$

$$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-2}, \qquad \frac{\partial^3 f(1,1)}{\partial x^3} = 0,$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = (2y-1)x^{y-2} + y(y-1)x^{y-2}\ln x, \qquad \frac{\partial^3 f(1,1)}{\partial x^2 \partial y} = 1,$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = 2x^{y-1}\ln x + yx^{y-1}(\ln x)^2, \qquad \frac{\partial^3 f(1,1)}{\partial y^2 \partial x} = 0,$$

$$\frac{\partial^3 f}{\partial y^3} = x^y (\ln x)^3, \qquad \frac{\partial^3 f(1,1)}{\partial y^3} = 0.$$

Natijada

$$f(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0) +$$

$$+\frac{1}{2}\left[\frac{\partial^2 f(x_0,y_0)}{\partial x^2}(x-x_0)^2 + 2\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}(x-x_0)(y-y_0) + \frac{\partial^2 f(x_0,y_0)}{\partial y^2}(y-y_0)^2\right] +$$

$$+\frac{1}{6}[\frac{\partial^3 f(x_0,y_0)}{\partial x^3}(x-x_0)^3 + 3\frac{\partial^3 f(x_0,y_0)}{\partial x^2 \partial y}(x-x_0)^2(y-y_0) +$$

$$+3\frac{\partial^3 f(x_0,y_0)}{\partial x \partial y^2}(x-x_0)\ (y-y_0)^2 + +\frac{\partial^3 f(x_0,y_0)}{\partial y^3}(y-y_0)^3] + R_3(f) = =$$

$$1 + 1(x-1) + 0\cdot(y-1) + \frac{1}{2}[0(x-1)^2 + 2\cdot 1(x-1)(y-1) + 0\cdot(y-1)^2] +$$

$$+\frac{1}{6}[0\cdot(x-1)^3 + 3\cdot 1\cdot(x-1)^2(y-1) + 3\cdot 0\cdot(x-1)(y-1)^2 + 0(y-1)^3] + R_3(f) =$$

$$= 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + R_3(f)$$

bo'ladi. Bu berilgan funksiyaning Teylor formulasidir.

13.9. Ko'p o'zgaruvchili funksiyaning ekstremum qiymatlari

$f(x_1, x_2, ..., x_m)$ funksiya ochiq $M (M \subset R^m)$ to'plamda berilgan bo'lib, $(x_1^0, x_2^0, ..., x_m^0) \in M$ bo'lsin.

Ta'rif. Agar $(x_1^0, x_2^0, ..., x_m^0)$ nuqtaning shunday U_δ atrofi:

$$U_\delta = \{(x_1, x_2, ..., x_m) \in R^m : \rho = \sqrt{\sum_{k=1}^{m}(x_k - x_k^0)^2} < \delta\} \subset M (\delta > 0)$$

mavjud bo'lsaki, $\forall (x_1, x_2, ..., x_m) \in U_\delta$ uchun

$$f(x_1,x_2,\ldots,x_m)\le f(x_1^0,x_2^0,\ldots,x_m^0) \quad \left(f(x_1,x_2,\ldots,x_m)\ge f(x_1^0,x_2^0,\ldots,x_m^0)\right)$$

bo‘lsa, $f(x_1,x_2,\ldots,x_m)$ funksiya $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqtada maksimumga (minimumga) ega deyiladi, $f(x_1^0,x_2^0,\ldots,x_m^0)$ qiymat esa $f(x_1,x_2,\ldots,x_m)$ funksiyaning ***maksimum (minimum) qiymati*** deyiladi. U

$$f(x_1^0,x_2^0,\ldots,x_m^0)=\max_{(x_1,\ldots,x_m)\in U_\delta}\{f(x_1,x_2,\ldots,x_m)\}$$

$$(f(x_1^0,x_2^0,\ldots,x_m^0)=\min_{(x_1,\ldots,x_m)\in U_\delta}\{f(x_1,x_2,\ldots,x_m)\}).$$

kabi belgilanadi.

Teorema. Agar $f(x_1,x_2,\ldots,x_m)$ funksiya $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqtada ekstremumga erishsa va shu nuqtada barcha $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\ldots,\frac{\partial f}{\partial x_m}$ xususiy hosilalarga ega bo‘lsa, u holda $\frac{\partial f(x_1^0,x_2^0,\ldots,x_m^0)}{\partial x_i}=0$, $i=1,2,\ldots,m$ bo‘ladi.

Teorema. $f(x_1,x_2,\ldots,x_m)$ funksiya $(x_1^0,x_2^0,\ldots,x_m^0)\in R^m$ nuqtaning biror $U_\delta(\delta>0)$ atrofida berilgan va ushbu shartlarni bajarsin:

1) $f(x_1,x_2,\ldots,x_m)$ funksiya U_δ da barcha o‘zgaruvchilari bo‘yicha birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega;

2) $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqta $f(x_1,\ldots,x_m)$ funksiyaning statsionar nuqtasi;

3) koeffitsientlari

$$a_{ik}=\frac{\partial^2 f(x_1^0,x_2^0,\ldots,x_m^0)}{\partial x_i\partial x_k} \quad (i,k=1,2,\ldots,m)$$

bo‘lgan

$$Q(\xi_1,\xi_2,\ldots,\xi_m)=\sum_{i,k=1}^{m}a_{ik}\xi_i\xi_k$$

kvadratik forma musbat (manfiy) aniqlangan.

U holda $f(x_1,x_2,\ldots,x_m)$ funksiya $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqtada minimumga (maksimumga) erishadi.

Agar kvadratik forma ishora saqlamasa, $f(x,y)$ funksiya $(x_1^0,x_2^0,\ldots,x_m^0)$ nuqtada ekstremumga erishmaydi.

Ikki o‘zgaruvchili funksiyalar uchun bu teorema quyidagicha bo‘ladi: $f(x,y)$ funksiya (x_0,y_0) nuqtaning

$$U_\delta=\left\{(x,y)\in R^2:\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\right\}(\delta>0)$$

atrofida berilgan va atrofda birinchi, ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo‘lsin. (x_0,y_0) nuqta $f(x,y)$ funksiyaning statsionar nuqtasi

$$\frac{\partial f(x_0,y_0)}{\partial x}=0, \quad \frac{\partial f(x_0,y_0)}{\partial y}=0$$

va

$$a_{11}=\frac{\partial^2 f(x_0,y_0)}{\partial x^2}, a_{12}=\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}, a_{22}=\frac{\partial^2 f(x_0,y_0)}{\partial y^2}$$

bo‘lsin.

1^0. Agar

$$a_{11}a_{22}-a_{12}^2>0 \quad \text{va} \quad a_{11}>0$$

bo‘lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada minimumga erishadi.

2^0. Agar

$$a_{11}a_{22}-a_{12}^2>0 \quad \text{va} \quad a_{11}<0$$

bo‘lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada maksimumga erishadi.

3^0. Agar

$$a_{11}a_{22}-a_{12}^2<0$$

bo‘lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada ekstremumga erishmaydi.

4^0. Agar

$$a_{11}a_{22}-a_{12}^2=0$$

bo‘lsa $f(x,y)$ funksiya (x_0,y_0) nuqtada ekstremumga erishishi ham, erishmasligi ham mumkin. Bu “shubhali” hol qo‘shimcha tekshirish talab qiladi.

Misol. Ushbu

$$f(x,y)=x^3+y^3-3axy \quad (a\neq 0)$$

funksiyani ekstremumga tekshiring.

Avvalo berilgan funksiyaning xususiy hosilalarini hisoblaymiz:

$$\frac{\partial f(x,y)}{\partial x}=3x^2-3ay,$$
$$\frac{\partial f(x,y)}{\partial y}=3y^2-3ax.$$

Ularni nolga tenglab,

$$\begin{cases}3x^2-3ay=0,\\ 3y^2-3ax=0.\end{cases}$$

sistemadan berilgan funksiyaning statsionar nuqtalari (0,0) hamda (a,a) ekanini topamiz.

Ravshanki,

$$\frac{\partial^2 f(x,y)}{\partial x^2}=6x, \quad \frac{\partial^2 f(x,y)}{\partial y^2}=6y, \quad \frac{\partial^2 f(x,y)}{\partial x\partial y}=-3a.$$

(a,a) nuqtada

$$a_{11}=\frac{\partial^2 f(a,a)}{\partial x^2}=6a,\quad a_{12}=\frac{\partial^2 f(a,a)}{\partial x\partial y}=-3a,\quad a_{22}=\frac{\partial^2 f(a,a)}{\partial y^2}=6a$$

bo‘lib,

$$a_{11}a_{22}-a_{12}^2=36a^2-9a^2=27a^2>0$$

bo‘ladi.

Demak, $a>0$ da $a_{11}>0$ bo‘lib, qaralayotgan funksiya (a,a) nuqtada minimumga, $a<0$ da $a_{11}<0$ bo‘lib, qaralayotgan funksiya (a,a) nuqtada maksimumga erishadi.

(0,0) nuqtada

$$a_{11}a_{22}-a_{12}^2=36\cdot 0-9a^2=-9a^2<0$$

bo‘lib, bu nuqtada funksiya ekstremumga erishmaydi.

$f(x,y,z)$ funksiya $(x_0,y_0,z_0)\in R^3$ nuqtaning biror U_δ atrofida berilgan va atrofda barcha birinchi, ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo‘lsin. (x_0,y_0,z_0) nuqta $f(x,y,z)$ funksiyaning statsionar nuqtasi

$$f'_x(x_0,y_0,z_0)=0,$$
$$f'_y(x_0,y_0,z_0)=0,$$
$$f'_z(x_0,y_0,z_0)=0,$$

va

$$A_1=a_{11},\quad A_2=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix},\quad A_3=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix},$$

bu yerda,

$$a_{11}=f''_{x^2}(x_0,y_0,z_0),\ a_{22}=f''_{y^2}(x_0,y_0,z_0),\ a_{33}=f''_{z^2}(x_0,y_0,z_0),$$
$$a_{12}=a_{21}=f''_{xy}(x_0,y_0,z_0),=f''_{yx}(x_0,y_0,z_0),\ a_{13}=a_{31}=f''_{xz}(x_0,y_0,z_0),=f''_{zx}(x_0,y_0,z_0),$$
$$a_{23}=a_{32}=f''_{yz}(x_0,y_0,z_0),=f''_{zy}(x_0,y_0,z_0),$$

bo‘lsin.

1) Agar $A_1>0,\quad A_2>0,\quad A_3>0$ bo‘lsa, $f(x,y,z)$ funksiya (x_0,y_0,z_0) nuqtada minimumga ega bo‘ladi.

2) Agar $A_1<0,\ A_2>0,\ A_3<0$ bo‘lsa, $f(x,y,z)$ funksiya (x_0,y_0,z_0) nuqtada maksimumga ega bo‘ladi.

3) Agar 1) va 2) guruhdagi shartlarning birortasi bajarilmasa qo‘shimcha tekshirish talab qilinadi.

14-§. OSHKORMAS FUNKSIYALAR

x va y o'zgaruvchilarning $F(x,y)$ funksiyasi uchun ushbu

$$F(x,y)=0$$

funksiyasini qaraylik.

Teorema. $F(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning biror $U_{h,k}((x_0,y_0))=\{(x,y)\in R^2 : x_0-h<x<x_0+h, y_0-k<y<y_0+k\}$ $(h>0, k>0)$ atrofida berilgan va quyidagi shartlarni qanoatlantirsin:

1) $U_{h,k}((x_0,y_0))$ da uzluksiz;

2) x o'zgaruvchining (x_0-h,x_0+h) oraliqdan olingan har bir tayin qiymatida y o'zgaruvchining funksiyasi sifatida o'suvchi;

3) $F(x_0,y_0)=0$.

U holda (x_0,y_0) nuqtaning shunday

$$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2 : x_0-\delta<x<x_0+\delta, y_0-\varepsilon<y<y_0+\varepsilon\}$$

atrofi $(0<\delta<h, 0<\varepsilon<k)$ topiladiki,

1) ixtiyoriy $x\in(x_0-\delta,x_0+\delta)$ uchun $F(x,y)=0$ tenglama yagona y yechimga $(y\in(y_0-\varepsilon,y_0+\varepsilon))$ ega, yani $F(x,y)=0$ tenglama yordamida

$$x\to y : F(x,y)=0$$

oshkormas ko'rinishdagi funksiya aniqlanadi.

2) $x=x_0$ bo'lganda unga mos kelgan $y=y_0$ bo'ladi, ya'ni $F(x_0,y_0)=0$;

3) oshkormas ko'rinishda aniqlangan

$$x\to y : F(x,y)=0$$

funksiya $(x_0-\delta,x_0+\delta)$ oraliqda uzluksiz bo'ladi.

Teorema. $F(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning biror atrofi $U(x_0,y_0)$ da aniqlangan bo'lib, quyidagi shartlarni qanoatlantirsin:

1^0. $F(x,y)$ funksiya U da n marta uzluksiz differensiallanuvchi ($n=1,2...$);

2^0. $F(x_0,y_0)=0$;;

3^0. $F'_y(x_0,y_0)\neq 0$.

U holda shunday $I\subset U(x_0,y_0)$ atrof va bu atrofda $f(x)$ funksiya mavjud bo'lib,

$$(I=I_x\times I_y; I_x=\{x\in R : |x-x_0|<\alpha\}, I_y=\{y\in R : |y-y_0|<\beta\})$$

ixtiyoriy $(x,y)\in I$ larda

1) $F(x,y)=0 \Leftrightarrow y=f(x)$;

2) $f(x)$ funksiya I_x da n-marta uzluksiz differensiallanuvchi va 1-tartibli hosila uchun

$$f^{`}(x) = -\frac{F_x^{`}(x, f(x))}{F_y^{`}(x, f(x))}$$

tenglik o‘rinli bo‘ladi.

Ushbu

$$\begin{cases} F_1 = F_1(x, y, u, v) = 0, \\ F_2 = F_2(x, y, u, v) = 0 \end{cases} \qquad (1)$$

tenglamalar sistemasini qaraylik.

Misol. Ushbu

$$F(x, y) = x^2 + y^2 - \ln y = 0 \quad (y > 0)$$

tenglama oshkormas funksiyani aniqlaydimi?

$y^2 - \ln y$ ayirma har doim musbat bo‘ladi:

$$y^2 - \ln y > 0.$$

Shu sababli x o‘zgaruvchining $(-\infty, \infty)$ dagi hech bir qiymatida

$$x^2 + y^2 - \ln y = 0$$

tenglik bajarilmaydi. Binobarin, berilgan tenglama oshkormas funksiyani aniqlamaydi.

M9. Ko’p o’zgaruvchili funksiyalar differensialiga doir mashqlar

1. Quyidagi funksiyalarning xususiy hosilalarini toping.

1) $z = 2x^2y - x^3 - y^3$

2) $z = \frac{x^2}{y}$

3) $z = (5xy^2 - x^3 + 6)^3$

4) $z = x^4t - t^4x + 5$

5) $z = \sin\frac{y}{x} \cdot \cos\frac{x}{y}$

6) $u = yx + xz + zy$

7) $u = \frac{y}{x} + \frac{x}{z} + \frac{z}{y}$

8) $u = \left(\frac{y}{x}\right)^z$

9) $f(x, y) = x^2 + y^2 + \sqrt{x^2 + y^2}$; $f'_x(4; 3)$, $f'_y(4; 3)$ larni toping.

10) $f(x, y) = \sqrt{xy + \frac{x}{y}}$; $f'_x(2; 1), f'_y(2; 1)$ larni toping.

11) $z = \ln(e^x + e^y)$ bo’lsa, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ o’rinli ekanligini isbotlang.

12) $z = x\ln\frac{y}{x}$ bo’lsa, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ tenglik o’rinli ekanligini ko’rsating.

2. Quyidagi funksiyalarning differensialini toping.

1) $f(x,y) = x^4y^5$
2) $z = \frac{y+1}{x+1}$
3) $f(x,y) = y\sqrt{x}$
4) $z = \sqrt{x^3 + y^3}$
5) $z = 2^{xy}$
6) $f\ (x,y) = \ln\sqrt{x^2 + y^2}$
7) $f\ (x,y) = e^{\sin x + \cos x}$
8) $z = \left(\frac{x}{y}\right)^y$
9) $f(x,y) = e^{\sin(xy)}$
10) $f(x,y) = arc\,\mathrm{tg}\frac{y}{x} + arc\,\mathrm{tg}\frac{x}{y}$

3. Quyidagi miqdorlarning taqribiy qiymatini toping.

1) $(0{,}95)^{1{,}04}$
2) $(1{,}05)^{2{,}96}$
3) $1{,}95^2 \cdot e^{0{,}1}$
4) $\sin 1{,}59 \cdot tg3{,}09$
5) $\sin 59° \cdot arctg0.05$

4. Quyidagi funksiyalarning ikkinchi tartibli xususiy hosilalarini toping.

1) $z = x^3 + x^2y + y^3$
2) $z = arctg\frac{y}{x}$
3) $z = arcctg\frac{x}{y}$
4) $\mathrm{z} = \sin\left(\frac{x}{a} - \frac{y}{b}\right)$; $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \cdot z = -\left(\frac{1}{a} - \frac{1}{b}\right)^2 \cdot z$ ekani tekshirilsin.
5) $z = arctg(2x - t)$; $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial t} = 0$ ekani tekshirilsin.
6) $z = \frac{x^2}{1-2y}$
7) $z = \frac{xy}{x-y}$
8) $z = e^{\frac{x}{y}}$

5. Quyidagi funksiyalarning ko'rsatilgan tartibdagi differensiallarini toping.

1) $f(x,y) = x^2 + y^2 + 5xy + 6{,}2$, d^2f
2) $f(x,y) = sinxy + coxy$, d^2f
3) $f(x,y) = 5x^3 + 5y^3 + 25xy$, d^3f
4) $f(x,y) = \sin(x^2 + y^2)$, d^3f
5) $f(x,y) = e^{xy}$, $d^{10}f$

6) $f(x,y) = \ln(xy),\ d^4 f$

6. Quyidagi funksiyalarni ekstremumga tekshiring.

1) $z = 2xy - 3x^2 - 2y^2 + 10$
2) $z = 4(x-y) - x^2 - y^2$
3) $z = x^2 + xy + y^2 + x - y + 1$
4) $z = x^3y^2(12 - x - y),\ (x > 0, y > 0)$
5) $z = x^3 + y^3 - 3xy$
6) $z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$
7) $z = x^2 + (y-1)^2$
8) $z = 2x^4 + y^4 - x^2 - 2y^2$

7. Quyidagi oshkormas ko'rinishda berilgan funksiyalarning birinchi va ikkinchi tartibli oshkormas hosilalarini toping.

1) $F(x,y) = x - y + \ln y = 0$
2) $F(x,y) = x^2 - 2xy + y^2 + x + y - 2 = 0$
3) $F(x,y) = xe^{2y} - y\ln x - 8 = 0$
4) $F(x,y) = e^y + x^2e^{-y} - 2x = 0$
5) $F(x,y) = x^2 lny - y^2 lnx = 0$

8. Quyidagi funksiyalarning berilgan sohadagi eng katta va eng kichik qiymatlarini toping.

1) $z = x^2 - y^2$
2) $z = x^2 + 2xy - 4x + 8y,\ 0 \le x \le 1,\ 0 \le y \le 2$
3) $z = e^{-x^2-y^2}(2x^2 + 3y^2),\ x^2 + y^2 \le 4$
4) $z = x^2 + y^2 - 12x + 16y$

15-§. FUNKSIONAL KETMA-KETLIKLAR VA QATORLAR

15.1. Funksional ketma-ketlik va qatorlarning yaqinlashuvchiligi

Faraz qilaylik, har bir natural $n \in N$ songa X to'plamda aniqlangan $f_n(x)$ funksiya mos kelsin. U holda

$$f_1(x), f_2(x), ..., f_n(x), ...$$

ketma-ketlik hosil bo'lib, bu ketma-ketlik ***funksional ketma-ketlik*** deyiladi. Funksional ketma-ketlik $\{f_n(x)\}$, uning umumiy hadi esa $f_n(x)$ kabi belgilanadi.

Ta'rif. Agar $\{f_n(x_0)\}$ sonlar ketma-ketligi ***yaqinlashuvchi (uzoqlashuvchi)*** bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi, x_0 nuqta esa bu funksional ketma-ketlikning yaqinlashish (uzoqlashish) nuqtasi deyiladi.

$\{f_n(x)\}$ funksional ketma-ketlikning barcha yaqinlashish nuqtalaridan iborat to'plam ketma-ketlikning yaqinlashish sohasi deyiladi. $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish sohasi M da aniqlangan ushbu

$$f : x \to \lim_{n\to\infty} f_n(x) \qquad (x \in M)$$

funksiya, $\{f_n(x)\}$ ketma-ketlikning limit funksiyasi deyiladi. Demak,

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in M).$$

15.2. Funksional ketma-ketlikning tekis yaqinlashuvchiligi

Biror $\{f_n(x)\}$:

$$f_1(x), f_2(x), ..., f_n(x), ...$$

funksional ketma-ketlik berilgan bo'lib, M esa bu funksional ketma-ketlikning yaqinlashish sohasi va $f(x)$ limit funksiyasi bo'lsin:

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in M).$$

Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ son olinganda ham shunday $n_0 \in N$ topilsaki, ixtiyoriy $n > n_0$ va barcha $x \in M$ lar uchun

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik M to'plamda $f(x)$ ***da tekis yaqinlashadi*** (funksional ketma-ketlik tekis yaqinlashuvchi) deyiladi.

Demak, bu holda ta’rifdagi n_0 natural son faqat ε ga bog‘liq bo‘lib, x larga bog‘liq bo‘lmaydi.

$\{f_n(x)\}$ funksional ketma-ketlik $f(x)$ ga tekis yaqinlashuvchiligi

$$f_n(x) \Rightarrow f(x) \qquad (x \in M)$$

kabi belgilanadi.

15.3. Funksional ketma-ketlikning notekis yaqinlashuvchilgi

Ta’rif. Agar ketma-ketlik ixtiyoriy $\varepsilon > 0$ va barcha x lar uchun umumiy n_0 topish mumkin bo‘lmasa, ya’ni ixtiyoriy $n \in N$ olinganda ham shunday ε_0 va $x_0 \in M$ topilsaki,

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilmasa, $\{f_n(x)\}$ funksional ketma-ketlik M to‘plamda $f(x)$ ga notekis yaqinlashadi deyiladi.

Bu holda n_0 natural son ε ga bog‘liq bo‘lishi bilan birga qaralayotgan x ga ham bog‘liq bo‘ladi.

Teorema. $\{f_n(x)\}$ funksional ketma-ketlikning M to‘plamda limit funksiya $f(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x\in M} |f_n(x) - f(x)| = 0$$

bo‘lishi zarur va yetarli.

Misol. Ushbu

$$f_n(x) = \frac{nx + x^2 + n^2}{x^2 + n^2} \qquad (0 \le x \le 1)$$

funksional ketma-ketlikni tekis yaqinlashuvchilikka tekshiring.

Bu ketma-ketlikning limit funksiyasi

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx + x^2 + n^2}{x^2 + n^2} = 1$$

bo‘ladi. Endi

$$|f_n(x) - f(x)| = \left|\frac{nx + x^2 + n^2}{x^2 + n^2} - 1\right| = \left|\frac{nx}{x^2 + n^2}\right| = \frac{nx}{x^2 + n^2}$$

ning supremumini topamiz. Ravshanki, $[0,1]$ da

$$\sup|f_n(x) - f(x)| = \sup \frac{nx}{x^2 + n^2} = \max \frac{nx}{x^2 + n^2}$$

bo‘ladi. Agar $x \in [0,1]$ va n>1 da

$$\left(\frac{nx}{x^2 + n^2}\right)' = \frac{n(x^2 + n^2) - nx \cdot 2x}{(x^2 + n^2)^2} = \frac{n^3 - nx^2}{(x^2 + n^2)^2} = \frac{n(n^2 - x^2)}{(x^2 + n^2)^2} > 0$$

ekanligini e'tiborga olsak, unda [0,1] da $\frac{nx}{x^2+n^2}$ ning o'suvchi ekanligini va $[0,1]$ da o'zining teng katta qiymatini $x=1$ da qabul qilishini aniqlaymiz.
Demak,

$$\max \frac{nx}{x^2+n^2} = \frac{n}{1+n^2}.$$

Shunday qilib, berilgan ketma-ketlik uchun

$$\sup_{0\le x\le 1} |f_n(x) - f(x)| = \frac{n}{1+n^2}$$

bo'lib, undan

$$\lim_{n\to\infty} \sup_{0\le x\le 1} |f_n(x) - f(x)| = 0$$

bo'lishi kelib chiqadi. Demak, berilgan ketma-ketlik $[0,1]$ da tekis yaqinlashuvchi.

15.4. Fundamental ketma-ketlik. Koshi teoremasi

Ta'rif. Agar ixtiyoriy $\varepsilon>0$ son olinganda ham shunday $n_0 \in N$ son topilsaki, $n>n_0, m>n_0$ bo'lganda ixtiyoriy $x\in X$ uchun

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X da ***fundamental*** ketma-ketlik deyiladi.

Teorema (Koshi teoremasi). $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi uchun X da fundamental bo'lishi zarur va yetarli.

15.5. Tekis yaqinlashuvchi funksional ketma-ketliklarning xossalari

M to'plamda $(M\subset R)$ biror $\{f_n(x)\}$ funksional ketma-ketlik berilgan bo'lib, uning limit funksiyasi $f(x)$ bo'lsin:

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x\in M)$$

1^0. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ ($n=1,2,3,...$) hadi M to'plamda uzluksiz bo'lib, M da tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ limit funksiya ham M to'plamda uzluksiz bo'ladi.

2^0. Agar $x\to x_0$ da $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ ($n=1,2,...$) hadi chekli

$$\lim_{x\to x_0} f_n(x) = a_n \qquad (n = 1, 2, 3, \ldots)$$

limitga ega bo'lib, M da tekis yaqinlashuvchi bo'lsa, u holda $\{a_n\}$ ketma-ketlik ham yaqinlashuvchi, uning limiti $a(a = \lim_{n\to\infty} a_n)$ esa $f(x)$ ning $x \to x_0$ dagi limitiga teng bo'ladi:

$$\lim_{x\to x_0} f(x) = a\,.$$

3⁰. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n = 1, 2, 3, \ldots)$ hadi $[a,b]$ da tekis yaqinlashuvchi bo'lsa, u holda

$$\int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \ldots, \int_a^b f_n(x)dx, \ldots$$

ketma-ketlik yaqinlashuvchi, uning limiti esa $\int_a^b f(x)dx$ ga teng bo'ladi:

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

4⁰. Agar $\{f_n(x)\}$ funksional ketma-ketlik har bir $f_n(x)$ $(n = 1, 2, 3, \ldots)$ hadi $[a,b]$ segmentda uzluksiz $f_n'(x)$ $(n = 1, 2, 3, \ldots)$ hosilaga ega bo'lib shu hosilalardan tuzilgan

$$f_1'(x), f_2'(x), \ldots, f_n'(x), \ldots$$

funksional ketma-ketlik $[a,b]$ da tekis yaqinlashuvchi bo'lsa, u holda limit funksiya $f(x)$ shu $[a,b]$ da $f'(x)$ hosilaga ega bo'lib, $\{f_n'(x)\}$ ketma-ketlikning limiti $f'(x)$ ga teng bo'ladi:

$$\lim_{n\to\infty}\left[\frac{d}{dx} f_n(x)\right] = f'(x) = \frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right].$$

15.6. Funksional qatorlar va ularning yaqinlashuvchiligi

X to'plamda $(X \subset R)$ biror

$$u_1(x), u_2(x), \ldots, u_n(x), \ldots$$

funksional ketma-ketlik berilgan bo'lsin. Ushbu

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

ifoda funksional qator deyiladi va $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \qquad (1)$$

Ta'rif. Agar $\sum_{n=1}^{\infty} u_n(x_0)(x_0 \in X)$ sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi, x_0 nuqta esa funksional qatorning ***yaqinlashish (uzoqlashish) nuqtasi*** deyiladi.

15.7. Funksional qatorning yaqinlashish sohasi

$\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning barcha yaqinlashish nuqtalaridan iborat to'plam funksional qatorning ***yaqinlashish sohasi*** deyiladi.

(1) funksional qatorning dastlabki hadlaridan tuzilgan ushbu

$$S_1(x) = u_1(x)$$
$$S_2(x) = u_1(x) + u_2(x),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

yig'indilar funksional qatorning qismiy yig'indilari deyiladi.

$\lim_{n\to\infty} S_n(x) = S(x)$ ga (1) funksional qatorning limiti deyiladi.

Misol. Ushbu

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots + x^{n-1} + \ldots$$

funksional qatorning yaqinlashish sohasini toping. Bu qatorning qismiy yig'indisi

$$S_n(x) = 1 + x + x^2 + \ldots + x^n = \begin{cases} \dfrac{1-x^n}{1-x}, agar \quad x \neq 1 \quad bo'lsa, \\ n, \quad agar \quad x = 1 \quad bo'lsa \end{cases}$$

bo'ladi. Unda

$$\forall x \in (-1,1) \quad uchun \quad \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$$
$$\forall x \in [1,+\infty) \quad uchun \quad \lim_{n\to\infty} S_n(x) = \infty,$$

ixtiyoriy $x \in (-\infty,-1]$ uchun $\{S_n(x)\}$ ketma-ketlik limitga ega emas.

Shunday qilib, berilgan funksional qatorning yaqinlashish sohasi $M(-1;1)$ intervaldan iborat ekan.

15.8. Funksional qatorning tekis yaqinlashuvchiligi

X to'plamda $(X \subset R)$ biror yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + .. + u_n(x) + \dots$$

funksional qator berilgan bo‘lib, uning yig‘indisi $S(x)$ bo‘lsin:

$$\lim_{n\to\infty} S_n(x) = S(x).$$

Ta’rif. Agar X to‘plamda $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning qismiy yig‘indilaridan iborat $\{S_n(x)\}$ funksional ketma-ketlik $S(x)$ ga tekis yaqinlashsa, u holda funksional qator X da tekis yaqinlashuvchi deyiladi.

$\{S_n(x)\}$ ketma-ketlik X da $S(x)$ ga notekis yaqinlashsa, unda $\sum_{n=1}^{\infty} u_n(x)$ funksional qator X da ***notekis yaqinlashuvchi*** deyiladi.

Teorema. $\sum_{n=1}^{\infty} u_n(x)$ funksional qator X da $S(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x\in X} |S_n(x) - S(x)| = \lim_{n\to\infty} \sup_{x\in X} \left| \sum_{k=n+1}^{\infty} u_k(x) \right| = 0$$

bo‘lishi zarur va yetarli.

Veyershtrass alomati. Agar

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$$

funksional qatorning har bir $u_n(x)$ $(n = 1, 2, 3, \dots)$ hadi X to‘plamda

$$|u_n(x)| \le c_n \quad (n = 1, 2, 3, \dots)$$

tengsizlikni qanoatlantirsa va

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \dots + c_n + \dots$$

sonli qator yaqinlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty} u_n(x)$ funksional qator X da ***tekis yaqinlashuvchi*** bo‘ladi.

Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2} = \frac{\sin 1^2 x}{1^2} + \frac{\sin 2^2 x}{2^2} + \dots + \frac{\sin n^2 x}{n^2} + \dots$$

funksional qatorni Veyershtrass alomatidan foydalanib tekis yaqinlashuvchiligini ko‘rsating.

Berilgan qatorning har bir

$$u_n(x) = \frac{\sin n^2 x}{n^2} \quad (n = 1,2,3\dots)$$

hadi uchun

$$|u_n(x)| = \left|\frac{\sin n^2 x}{n^2}\right| \le \frac{1}{n^2}$$

bo‘ladi va ravshanki,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

sonli qator yaqinlashuvchi. Veyershtrass alomatiga ko‘ra berilgan funksional qator $(-\infty, +\infty)$ da tekis yaqinlashuvchi bo‘ladi.

Teorema (***Koshi teoremasi***). $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning X da tekis yaqinlashuvchi bo‘lishi uchun uning qismiy yig‘indilari ketma-ketlikning X da fundamental bo‘lishi zarur va yetarli.

15.9. Tekis yaqinlashuvchi funksional qatorlarning xossalari

1^0. Funksional qator yig‘indisining uzluksizligi. Agar $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning hadlari X to‘plamda uzluksiz bo‘lib, bu funksional qator X da tekis yaqinlashuvchi bo‘lsa, u holda qatorning yig‘indisi $S(x)$ ham X da uzluksiz bo‘ladi.

2^0. Funksional qatorlarda hadma-had limitga o‘tish. Agar $x \to x_0$ da $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning har bir $u_n(x)$ $(n = 1, 2, 3, \ldots)$ hadi chekli

$$\lim_{x \to x_0} u_n(x) = c_n \qquad (n = 1, 2, 3, \ldots)$$

limitga ega bo‘lib, bu qator X da tekis yaqinlashuvchi bo‘lsa, u holda

$$\sum_{n=1}^{\infty} c_n$$

qator ham yaqinlashuvchi, uning yig‘indisi c esa $S(x)$ ning $x \to x_0$ dagi limiti

$$\lim_{x \to x_0} S(x) = c$$

ga teng bo‘ladi:

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x).$$

3^0. Funksional qatorni hadma-had integrallash. Agar $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning har bir $u_n(x)$ $(n = 1, 2, 3, \ldots)$ hadi $[a, b]$ segmentda uzluksiz bo‘lib, qator shu segmentda tekis yaqinlashuvchi bo‘lsa, u holda qator hadlarining integrallaridan tuzilgan

$$\int_a^b u_1(x)dx + \int_a^b u_2(x)dx + \ldots + \int_a^b u_n(x)dx + \ldots$$

qator ham yaqinlashuvchi, uning yig‘indisi esa $\int_a^b S(x)dx$ ga teng bo‘ladi:

$$\int_a^b (\sum_{n=1}^{\infty} u_n(x))dx = \sum_{n=1}^{\infty} (\int_a^b u_n(x)dx).$$

4^0. Funksional qatorni hadma-had differensiallash. Agar $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning har bir $u_n(x)$ hadi ($n = 1, 2, 3, \ldots$) $[a,b]$ segmentda uzluksiz $u_n'(x)$ ($n = 1, 2, 3, \ldots$) hosilaga ega bo‘lib, bu hosilalardan tuzilgan $\sum_{n=1}^{\infty} u_n^{`}(x)$ funksional qator $[a,b]$ da tekis yaqinlashuvchi bo‘lsa, u holda berilgan funksional qatorning yig‘indisi $S(x)$ shu $[a,b]$ da $s^{`}(x)$ hosilaga ega va

$$S^{`}(x) = \sum_{n=1}^{\infty} u_n^{`}(x)$$

bo‘ladi:

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

M10. Funksional ketma-ketliklar va qatorlarga doir mashqlar

1. Quyidagi funksional ketma-ketliklarning limit funksiyalarini toping.

1) $f_n(x) = x^n, \; -\infty < x < +\infty$

2) $f_n(x) = x^n - x^{2n}, \; 0 \le x \le 1$

3) $f_n(x) = nx^2 \sin\frac{x}{\sqrt{n}}, \; -\infty < x < +\infty$

4) $f_n(x) = \cos^n \frac{x}{\sqrt{n}}, \; -\infty < x < +\infty$

5) $f_n(x) = \frac{1 + x^{2n}}{2 + x^{4n}}, \; -\infty < x < +\infty$

6) $f_n(x) = \frac{1}{x^2 + n}, \; -\infty < x < +\infty$

7) $f_n(x) = n^2 x(1 - x^2)^n, \; 0 \le x \le 1$

8) $f_n(x) = \left(\frac{x + n}{2x + n}\right)^{2(x+n)}$

9) $f_n(x) = n^2(\sqrt[n]{x} - \sqrt[n+1]{x}), \; x > 0$

10) $f_n(x) = \left(1 + \frac{x}{n}\right)^n, \; -\infty < x < +\infty$

2. Quyidagi funksional ketma-ketliklarni tekis yaqinlashuvchiligini ko’rsating.

1) $f_n(x) = \frac{\sin nx}{n}, \; -\infty < x < +\infty$

2) $f_n(x) = \frac{nx}{1 + n + x}, \; 0 \le x \le 1$

3) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}, \; -\infty < x < +\infty$

4) $f_n(x) = \frac{nx^2}{n + x}, \; 1 \le x < +\infty$

5) $f_n(x) = xe^{-nx}, \; 0 \le x < +\infty$

6) $f_n(x) = \ln\left(x^2 + \frac{1}{n}\right), \; 1 \le x < +\infty$

7) $f_n(x) = e^{-(x-n)^2}, \ -1 \le x \le 1$

8) $f_n(x) = \arcsin \dfrac{x^n}{1+x^n}, \ 0 \le x \le \dfrac{1}{2}$

3. Quyidagi funksional ketma-ketliklarni tekis hamda notekis yaqinlashishga tekshiring.

1) $f_n(x) = x^n - x^{n+1}, \ 0 \le x \le 1$

2) $f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}} - \sqrt{x}\right), \ 0 < x < +\infty$

3) $f_n(x) = \dfrac{\sin nx}{n}, \ -\infty < x < +\infty$

4) $f_n(x) = \sqrt[n]{x \sin x}, \ 0 \le x \le \pi$

5) $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}, \ -2 \le x \le 2$

6) $f_n(x) = \dfrac{x}{n} \ln \dfrac{x}{n}, \ 0 < x < 1$

7) $f_n(x) = nxe^{-nx^2}, \ 0 \le x \le 1$

4. Funksional qatorlarning yaqinlashish sohasini toping.

1) $\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)x^{2n-1}}$

2) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{(2n-1)} \left(\dfrac{1-x}{1+x}\right)^n$

3) $\sum\limits_{n=0}^{\infty} 2^n \sin \dfrac{x}{3^n}$

4) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{(x-2)^n}$

5) $\sum\limits_{n=1}^{\infty} \dfrac{1}{(x+2)^n}$

6) $\sum\limits_{n=1}^{\infty} n\sqrt[3]{\sin^n x}$

7) $\sum\limits_{n=1}^{\infty} \dfrac{x^n(1-x^n)}{n}$

8) $\sum\limits_{n=1}^{\infty} xe^{nx}$

5. Quyidagi funksional qatorlarning absolyut yaqinlashish sohalarini toping.

1) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{\pi x}{n}$

2) $\sum\limits_{n=1}^{\infty} n^2 \left(\dfrac{2x-3}{4}\right)^n$

3) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!x^n}$

4) $\sum\limits_{n=0}^{\infty} (-1)^{n+1} e^{-n \sin x}$

5) $\sum\limits_{n=1}^{\infty} \left(\dfrac{x^2}{n} + x\right)^n$

6) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{3^{n-1}\sqrt{n}}$

7) $\sum\limits_{n=1}^{\infty} n^{-\ln x^2}$

6. Quyidagi funksional qatorlarni ko'rsatilgan oraliqda tekis yaqinlashuvchiligini isbotlang.

1) $\sum\limits_{n=1}^{\infty} \dfrac{1}{x^2+n^2}, \ -\infty < x < +\infty$

2) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n^2 x}, \ 1 < x < +\infty$

3) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{x+\sqrt{n}}, \ 0 \le x < +\infty$

4) $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{2^n}, \ -\infty < x < +\infty$

5) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n\sqrt{n}+x^2}, \ -\infty < x < +\infty$

6) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^{2n}}{n}, \ -1 < x < 1$

7. Quyidagi funksional qatorlarni ko'rsatilgan oraliqda tekis yaqinlashuvchiligini Veyrshtrass alomatidan foydalanib isbotlang.

1) $\sum_{n=1}^{\infty}\frac{x^n}{n^2},\ -1\le x\le 1$ 2) $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+2^n},\ -2<x<+\infty$

3) $\sum_{n=1}^{\infty}e^{-\sqrt{nx}},\ -1\le x<+\infty$ 4) $\sum_{n=1}^{\infty}\frac{\sin nx}{n\sqrt{n}},\ -\infty<x<+\infty$

5) $\sum_{n=1}^{\infty}\frac{x}{1+n^4x^2},\ 0\le x<+\infty$ 6) $\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2},\ -\infty<x<+\infty$

7) $\sum_{n=1}^{\infty}\frac{(x-1)^n}{(3n+1)\cdot 3^n},\ -1\le x\le 3$ 8) $\sum_{n=1}^{\infty}\sin^2\frac{\sqrt{x}}{1+n^2x},\ 0\le x<+\infty$

8. Quyidagi funksional qatorlar yig'indisini uzluksizlikka tekshiring.

1) $\sum_{n=1}^{\infty}\frac{x^n}{2^n(1+x^{2n})}$ 2) $\sum_{n=1}^{\infty}x^2e^{-nx},\ 0\le x\le 1$

3) $\sum_{n=1}^{\infty}\frac{1}{n^4+n^2x^2}$ 4) $\frac{1}{1+x}+\sum_{n=1}^{\infty}\frac{x}{[1+(n+1)x](1+nx)}$

5) $x+\sum_{n=1}^{\infty}\left(x^{n+1}-x^n\right),\ 0\le x\le 1$ 6) $\frac{1}{1+x}+\sum_{n=1}^{\infty}\left(\frac{1}{x+n+1}-\frac{1}{x+n}\right),\ 0\le x\le 1$

9. Quyidagi funksional qatorlarni yaqinlashish sohasida hadlab integrallash mumkinmi?.

1) $\sum_{n=1}^{\infty}\frac{\cos nx}{n^3}$ 2) $\sum_{n=1}^{\infty}\frac{x}{(1+x^2)^n}$

3) $\sum_{n=1}^{\infty}\left(2^{2n}-x^{2n-2}\right)$

10. Quyidagi funksional qatorlarni yaqinlashish sohasida hadlab differensiallash mumkinmi?

1) $\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}$ 2) $\sum_{n=1}^{\infty}\frac{\sin 3^n\pi x}{3^n}$ 3) $\sum_{n=1}^{\infty}e^{-(x-n)^2}$

11. Quyidagi limitlarni toping.

1) $\lim_{x\to\frac{1}{2}}\sum_{n=1}^{\infty}x^{n-1}$ 2) $\lim_{x\to 1-0}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{2^{n-1}}$

3) $\lim_{x\to\frac{\pi}{2}}\sum_{n=1}^{\infty}\left(\frac{\sin nx}{\sqrt{n}}-\frac{\sin(n+1)x}{\sqrt{n+1}}\right)$ 4) $\lim_{x\to\infty}\sum_{n=1}^{\infty}\frac{x^2}{1+n^2x^2}$

16-§. DARAJALI QATORLAR

Ushbu

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

yoki umumiyroq

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n + \ldots$$

qatorlar (bunda $a_0, a_1, \ldots, a_n, \ldots va - x_0$ -o'zgarmas haqiqiy sonlar) ***darajali qatorlar*** deyiladi.

Teorema (***Abel teoremasi***). Agar $\sum_{n=0}^{\infty} a_n x^n$ darajali qator x ning $x = x_0$ $(x_0 \neq 0)$ qiymatida yaqinlashuvchi bo'lsa, x ning

$$|x| < |x_0|$$

tengsizlikni qanoatlantiruvchi barcha qiymatlarida darajali qator absolyut yaqinlashuvchi bo'ladi.

Teorema. Agar $\sum_{n=0}^{\infty} a_n x^n$ darajali qator x ning ba`zi $(x \neq 0)$ qiymatlarida yaqinlashuvchi, ba`zi qiymatlarida uzoqlashuvchi bo'lsa, u holda shunday yagona r $(r > 0)$ son topiladiki, $\sum_{n=0}^{\infty} a_n x^n$ darajali qator x ning $|x| < r$ tengsizlikni qanoatlantiruvchi qiymatlarida absolyut yaqinlashuvchi, $|x| > r$ tengsizlikni qanoatlantiruvchi qiymatlarida esa uzoqlashuvchi bo'ladi.

16.1. Darajali qatorning yaqinlashish radiusi

Ta'rif. Yuqoridagi teoremadan r soni $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning ***yaqinlashish radiusi***, $(-r, r)$ interval esa darajali qatorning ***yaqinlashish intervali*** deyiladi.

Eslatma. $x = \pm r$ nuqtalarda $\sum_{n=0}^{\infty} a_n x^n$ darajali qator yaqinlashishi ham mumkin, uzoqlashishi ham mumkin.

Teorema (*Koshi-Adamar teoremasi*). Berilgan $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi

$$r=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} \qquad (\overline{\lim\limits_{n\to\infty}}\ -\text{yuqori limit}) \qquad (1)$$

bo‘ladi.

Eslatma. Agar $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=0$ bo‘lsa, $r=+\infty$; agar $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=+\infty$ bo‘lsa, $r=0$ bo‘ladi.

Eslatma. $\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$ darajali qatorning yaqinlashish intervali (x_0-r,x_0+r) bo‘ladi. Bunda r ushbu $\sum\limits_{n=0}^{\infty}a_nx^n$ qatorning yaqinlashish radiusi.

Misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{x^n}{2^{\sqrt{n}}}=\frac{x}{2}+\frac{x^2}{2^{\sqrt{2}}}+\ldots+\frac{x^n}{2^{\sqrt{n}}}+\ldots$$

darajali qatorning yaqinlashish radiusini, yaqinlashish intervalini va yaqinlashish sohasini toping.

Bu darajali qatorning yaqinlashish radiusini (1) ga ko‘ra topamiz:

$$r=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}}=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\frac{1}{2^{\sqrt{n}}}}}=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{2^{\sqrt{n}}}=1$$

Demak, berilgan darajali qatorning yaqinlashish radiusi $r=1$, yaqinlashish intervali esa (-1,1) bo‘ladi. $x=\pm r=\pm1$ da darajali qator mos ravishda

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{\sqrt{n}}},\quad \sum_{n=1}^{\infty}\frac{1}{2^{\sqrt{n}}}$$

sonli qatorlatga aylanadi. Bu qatorlarning yaqinlashuvchiligi ma`lum. Demak, berilgan darajali qatorning yaqinlashish sohasi [-1,1] segmentdan iborat.

16.2. Darajali qatorlarning xossalari

Biror

$$\sum_{n=1}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \qquad (1)$$

berilgan bo‘lsin.

1⁰. Agar (1) qatorning yaqinlashish radiusi r ($r>0$) bo‘lsa, u holda qator $[-c,c]$ ($0<c<r$) da tekis yaqinlashuvchi bo‘ladi.

2⁰. Agar (1) qatorning yaqinlashish radiusi r bo‘lsa, u holda uning

$$S(x)=\sum_{n=0}^{\infty}a_n x^n$$

yig'indisi $(-r,r)$ da uzluksiz funksiya bo'ladi.

3^0**.** Agar (1) darajali qatorning yaqinlashish radiusi r bo'lib, qator $x=r$ $(x=-r)$ nuqtalarda yaqinlashuvchi bo'lsa, u holda qatorning yig'indisi $S(x)$ funksiya $x=r$ $(x=-r)$ nuqtada chapdan (o'ngdan) uzluksiz bo'ladi.

4^0**.** Agar (1) qatorning yaqinlashish radiusi r bo'lsa, uni $[a,b]$ ($[a,b]\subset(-r,r)$) oraliqda hadlab integrallash mumkin.

5^0**.** Agar (1) qatorning yaqinlashish radiusi r bo'lsa, uni $(-r,r)$ da hadlab differensiallash mumkin.

Misol. Ushbu

$$\sum_{n=1}^{\infty}nx^n$$

funksional qatorning yig'indisini toping.

Ma`lumki,

$$\sum_{n=1}^{\infty}x^n$$

darajali qator (-1,1) da yaqinlashuvchi va uning yig'indisi $\frac{x}{1-x}$ ga teng:

$$\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}.$$

Bu qatorni hadlab differensiallab topamiz:

$$\frac{d}{dx}\sum_{n=1}^{\infty}x^n=\frac{d}{dx}\left(\frac{x}{1-x}\right)\Rightarrow\sum_{n=1}^{\infty}nx^{n-1}=\frac{1-x-x(-1)}{(1-x)^2}\Rightarrow\sum_{n=1}^{\infty}nx^{n-1}=\frac{x}{(1-x)^2}.$$

16.3. Teylor qatori. Funksiyalarni darajali qatorlarga yoyish

$f(x)$ funksiya $x_0(x_0\in R)$ nuqtaning biror $U_\delta(x_0)=\{x\in R: x_0-\delta<x<x_0+\delta;\delta>0\}$ atrofida berilgan bo'lib, shu atrofda istalgan tartibdagi hosilaga ega bo'lsin. Ushbu

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots,$$

bu yerda $f^{(0)}(x_0)\equiv f(x_0)$, darajali qator $f(x)$ funksiyaning ***Teylor qato***ri deyiladi. Xususan, $x_0=0$ da qator quyidagicha bo'ladi:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\frac{f'(0)}{1!}x+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

(odatda bu qatorni ***Makloren qatori*** ham deyiladi).

Teorema. $f(x)$ funksiya biror $(-r,r)$ $(r>0)$ oraliqda istalgan tartibdagi hosilaga ega bo‘lib, uning $x=0$ nuqtadagi Teylor qatori

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

bo‘lsin. Bu qator $(-r,r)$ da $f(x)$ ga yaqinlashishi uchun $f(x)$ funksiya Teylor formulasi

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+r_n(x)$$

ning qoldiq hadi barcha $x\in(-r,r)$ da nolga intilishi

$$\lim_{n\to\infty} r_n(x)=0$$

zarur va yetarli.

Teorema. $f(x)$ funksiya biror $(-r,r)$ oraliqda istalgan tartibdagi hosilaga ega bo‘lsin. Agar shunday o‘zgarmas $M>0$ soni topilsaki, barcha $x\in(-r,r)$ hamda barcha n $(n=1,2,\ldots)$ uchun

$$\left|f^{(n)}(x)\right|<M$$

tengsizlik bajarilsa, u holda $(-r,r)$ oraliqda $f(x)$ funksiya Teylor qatoriga yoyiladi, ya’ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots$$

bo‘ladi.

Misol. Ushbu

$$f(x)=\ln\frac{1+x}{1-x}$$

funksiyani Makloren qatoriga yoying.

Ma`lumki, $x\in(-1;1)$ da

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

bo‘ladi. Bunda x ni $-x$ ga almashtirib, topamiz:

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots-\frac{x^n}{n}-\ldots.$$

Natijada

$$\ln(1+x):\ln(1-x)=2x+\frac{2x^2}{3}+\ldots+\frac{2x^{2n-1}}{2n-1}+\ldots$$

bo‘ladi. Keyingi qatorning (-1,1) da yaqinlashuvchiligi ravshan. Demak,

$$\ln\frac{1+x}{1-x}=2x+\frac{2x^2}{3}+\ldots+\frac{2x^{2n-1}}{2n-1}+\ldots$$

11. Darajali qatorlarga doir mashqlar

1. Quyidagi darajali qatorlarning yaqinlashish radiusi, yaqinlashish intervali hamda yaqinlashish sohalari topilsin.

1) $\frac{1^2}{2^2}+\frac{2^2}{3^2}\cdot\frac{x^2}{2}+\frac{3^2}{4^2}\cdot\frac{x^4}{2^2}+\ldots+\frac{n^2}{(n+1)^2}\cdot\frac{x^{2n}}{2^n}+\ldots$

2) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ 3) $\sum_{n=1}^{\infty}n!x^n$

4) $\sum_{n=1}^{\infty}(-2)^n x^{2n}$ 5) $\sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}(x+1)^n$

6) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$ 7) $\sum_{n=1}^{\infty}\frac{(x+3)^{n^2}}{n^n}$

8) $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{2n+1}$ 9) $\sum_{n=1}^{\infty}2^{\ln n}x^n$

10) $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)x^n$ 11) $\sum_{n=1}^{\infty}\frac{x^{n^2}}{2^n}$

2. Quyidagi darajali qatorlarning yig'indilarini hadlab differensiallash va integrallash yordamida toping.

1) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ 2) $\sum_{n=1}^{\infty}(n+1)x^n$

3) $\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}$ 4) $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

5) $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$ 6) $\sum_{n=0}^{\infty}(n+1)x^n$

7) $\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)x^{2n-2}$ 8) $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$

3. Quyidagi funksiyalarni ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va bu qatorlarning yaqinlashish radiusini toping.

1) $f(x)=\sin^4 x,\ x_0=\frac{\pi}{4}$ 2) $f(x)=\cos^4 x,\ x_0=-\frac{\pi}{2}$

3) $f(x)=\ln(x^2+2x+2),\ x_0=-1$ 4) $f(x)=\frac{1}{x^2-5x+6},\ x_0=1$

5) $f(x)=\frac{1}{x},\ x=-2$ 6) $f(x)=e^x,\ x_0=\frac{\pi}{4}$

7) $f(x)=\sqrt{x},\ x_0=4$

4. Quyidagi funksiyalarni turli usullardan foydalanib Makloren qatoriga yoying va bu qatorni yaqinlashish radiusini toping

1) $f(x)=\ln\frac{1+x}{1-x}$ 2) $f(x)=(1+x)e^{-x}$

3) $f(x)=\arcsin\frac{x}{\sqrt{1+x^2}}$ 4) $f(x)=\sin^2 x\cos^2 x$

5) $f(x)=\arccos(1-2x^2)$ 6) $f(x)=\dfrac{x^2-3x+1}{x^2-5x+6}$

7) $f(x)=\arccos\dfrac{1-x}{1+x}$

17-§. XOSMAS INTEGRALLAR

17.1. Cheksiz oraliq bo‘yicha xosmas integrallar

$f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo‘lib, bu oraliqning istalgan $[a,t]$ $(a<t<+\infty)$ qismida integrallanuvchi, ya’ni ixtiyoriy t $(t>a)$ uchun ushbu

$$F(t)=\int_a^t f(x)dx$$

integral mavjud bo‘lsin.

Ta’rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud bo‘lsa, bu limit $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo‘yicha xosmas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_a^{+\infty} f(x)dx=\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}\int_a^t f(x)dx \qquad (1)$$

17.2. Xosmas integrallarning yaqinlashuvchiligi

Ta’rif. Agar $t\to+\infty$ da $F(x)$ funksiyaning limiti mavjud va chekli bo‘lsa, (1) xosmas integral yaqinlashuvchi deyiladi, $f(x)$ esa cheksiz $[a,+\infty)$ ***oraliqda integrallanuvchi*** deyiladi.

Agar $t\to+\infty$ da $F(x)$ funksiyaning limiti cheksiz bo‘lsa yoki limit mavjud bo‘lmasa, (1) xosmas integral uzoqlashuvchi deyiladi.

Misol. Ushbu

$$I=\int_0^{+\infty} xe^{-x^2}dx$$

xosmas integralning yaqinlashuvchiligini aniqlang va qiymatini toping.

$$\int_0^{+\infty} xe^{-x^2}dx=\lim_{t\to+\infty}\int_0^t xe^{-x^2}dx$$

bo‘lib,

$$F(t)=\int_0^t xe^{-x^2}dx=\int_0^t e^{-x^2}\frac{1}{2}dx^2=-\frac{1}{2}e^{-x^2}\Big|_0^t=-\frac{1}{2}e^{-t^2}+\frac{1}{2}$$

bo‘lganligidan

$$\lim_{t\to+\infty}F(t)=\frac{1}{2}$$

bo‘lishi kelib chiqadi.

Demak, berilgan xosmas integral yaqinlashuvchi va

$$\int_0^{+\infty} xe^{-x^2}dx=\frac{1}{2}.$$

17.3. Yaqinlashuvchi xosmas integrallarning xossalari. Asosiy formulalar

1⁰. Agar $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ xosmas integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty}[\alpha f(x)\pm\beta g(x)]dx$$

xosmas integral ham yaqinlashuvchi bo‘lib,

$$\int_a^{+\infty}[\alpha f(x)\pm\beta g(x)]dx=\alpha\int_a^{+\infty} f(x)dx\pm\beta\int_a^{+\infty} g(x)dx$$

bo‘ladi, bunda α,β -o‘zgarmas sonlar.

2⁰. Agar ixtiyoriy $x\in[a,+\infty)$ uchun $f(x)\le g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx\le\int_a^{+\infty} g(x)dx$$

bo‘ladi.

3⁰. ***Nyuton-Leybnis formulasi.*** $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz bo‘lib, $F(x)$ uning shu oraliqdagi boshlang‘ich funksiyasi bo‘lsin ($F'(x)=f(x)$). Unda

$$\int_a^{+\infty} f(x)dx=F(x)\Big|_a^{+\infty}=F(+\infty)-F(a) \qquad (2)$$

bo‘ladi. Bu yerda

$$F(+\infty)=\lim_{t\to+\infty}F(t).$$

4⁰. O‘zgaruvchini almashtirish formulasi. $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz, $\varphi(t)$ funksiya esa $[\alpha,\beta)$ da uzluksiz differensiallanuvchi funksiya bo‘lib,

$$a=\varphi(\alpha)\leq\varphi(t)<\lim_{t\to\beta-0}\varphi(t)=+\infty$$

bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx=\int_a^{+\infty} f(\varphi(t))\varphi'(t)dt$$

bo‘ladi.

5⁰. Bo‘laklab integrallash formulasi. Agar $u=u(x)$ va $v=v(x)$ funksiyalar $[a,+\infty)$ da uzluksiz differensiallanuvchi bo‘lib, $\lim\limits_{x\to+\infty}(uv)$ mavjud bo‘lsa, u holda

$$\int_a^{+\infty} udv=uv\Big|_a^{+\infty}-\int_a^{+\infty} vdu$$

bo‘ladi. Bu yerda

$$uv\Big|_a^{+\infty}=\lim_{x\to+\infty}(uv)-u(a)v(a)\,.$$

Misol. Ushbu

$$\int_2^{+\infty}\frac{dx}{x^2+x-2}$$

integralni hisoblang. Ma`lumki,

$$f(x)=\frac{1}{x^2+x-2}=\frac{1}{3}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)$$

bo‘lib,

$$F(x)=\frac{1}{3}\ln\frac{x-1}{x+2}$$

$f(x)$ ning boshlang‘ich funksiyasidir. Nyuton-Leybnis formulasiga ko‘ra topamiz:

$$\int_2^{+\infty}\frac{dx}{x^2+x-2}=\frac{1}{3}\ln\frac{x-1}{x+2}\Big|_2^{+\infty}=\frac{2}{3}\ln 2$$

17.4. Xosmas integrallarning yaqinlashuvchiligi haqida teoremalar

Teorema. $f(x)$ funksiyaning xosmas integrali $\int_a^{+\infty} f(x)dx$ yaqinlashuvchi bo‘lishi uchun, ixtiyoriy $t\in(a,+\infty)$ da $F(t)=\int_a^t f(x)dx\leq C$ ($C=const$) bo‘lishi zarur va yetarli.

Teorema. $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da berilgan bo‘lib, $\forall x\in[a,+\infty)$ da

$$0\le f(x)\le g(x)$$

bo‘lsin. U holda $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsa, $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo‘ladi; $\int_a^{+\infty} f(x)dx$ uzoqlashuvchi bo‘lsa, $\int_a^{+\infty} g(x)dx$ ham uzoqlashuvchi bo‘ladi.

Misol. Ushbu

$$\int_0^{+\infty} e^{-x^2}dx$$

integralning yaqinlashuvchiligini ko‘rsating. Ravshanki, ixtiyoriy $x\ge1$ uchun

$$e^{-x^2}\le\frac{1}{x^2}$$

bo‘ladi. Unda

$$\int_1^{+\infty}\frac{dx}{x^2}$$

ning yaqinlashuvchi bo‘lishini e’tiborga olib, yuqoridagi teoremaga binoan

$$\int_1^{+\infty} e^{-x^2}dx$$

ning ham yaqinlashuvchi ekanini topamiz. Ravshanki,

$$\int_0^{+\infty} e^{-x^2}dx=\int_0^{1} e^{-x^2}dx+\int_1^{+\infty} e^{-x^2}dx,$$

bu yerda $\int_0^{1} e^{-x^2}dx$ ham yaqinlashuvchi. Demak, berilgan xosmas integral yaqinlashuvchi.

Teorema. $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da $f(x)\ge0, g(x)\ge0$ bo‘lib,

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k \qquad (0\le k\le+\infty)$$

bo‘lsin. Agar $k<+\infty$ va $\int_a^{+\infty} g(x)dx$ integral yaqinlashuvchi bo‘lsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashuvchi bo‘ladi. Agar $k>0$ va $\int_a^{+\infty} g(x)dx$ integral uzoqlashuvchi bo‘lsa, $\int_a^{+\infty} f(x)dx$ integral ham uzoqlashuvchi bo‘ladi.

Teorema. Agar $x \to +\infty$ da $f(x)$ funksiya $\frac{1}{x}$ ga nisbatan $\alpha(\alpha > 0)$ tartibli cheksiz kichik bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ xosmas integral $\alpha > 1$ bo‘lganda yaqinlashuvchi, $\alpha \le 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

Teorema (*Koshi teoremasi*). Quyidagi

$$\int_a^{+\infty} f(x)dx$$

xosmas integralning yaqinlashuvchi bo‘lishi uchun, ixtiyoriy $\varepsilon > 0$ son olinganda ham, shunday $t_0 (t_0 > a)$ son topilib, $t' > t_0, t'' > t_0$ bo‘lgan ixtiyoriy t', t'' lar uchun

$$|F(t'') - F(t')| = \left| \int_a^{t''} f(x)dx - \int_a^{t'} f(x)dx \right| = \left| \int_{t'}^{t''} f(x)dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

17.5. Xosmas integralning absolyut yaqinlashuvchiligi. Dirixle alomati

Ta’rif. Agar $\int_a^{+\infty} |f(x)|dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ absolyut yaqinlashuvchi integral deyiladi, $f(x)$ funksiya esa $[a, +\infty)$ da absolyut integrallanuvchi funksiya deyiladi.

Teorema (*Dirixle alomati*). $f(x)$ va $g(x)$ funksiyalar $[a, +\infty)$ oraliqda berilgan bo‘lib, ular quyidagi shartlarni bajarsin:

1) $f(x)$ funksiya $[a, +\infty)$ da uzluksiz va uning shu oraliqdagi boshlang‘ich $F(x)$ ($F'(x) = f(x)$) funksiyasi chegaralangan,

2) $g(x)$ funksiya $[a, +\infty)$ da $g'(x)$ uzluksiz hosilaga ega funksiya,

3) $g(x)$ funksiya $[a, +\infty)$ da kamayuvchi,

4) $\lim_{x \to +\infty} g(x) = 0$.

U holda

$$\int_a^{+\infty} f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

17.6. Chegaralanmagan funksiyaning xosmas integrallari va ularning yaqinlashuvchiligi tushunchalari

$f(x)$ funksiya $[a,b)$ yarim intervalda berilgan bo‘lib, b nuqta $f(x)$ ning maxsus nuqtasi bo‘lsin. Bu funksiya $[a,b)$ ning istalgan $[a,t]$ ($a<t<b$) qismida integrallanuvchi, ya’ni ixtiyoriy t ($a<t<b$) uchun ushbu

$$F(t)=\int_a^t f(x)dx$$

integral mavjud bo‘lsin.

Agar $t\to b-0$ da $F(t)$ funksiyaning limiti

$$\lim_{t\to b-0}F(t)$$

mavjud bo‘lsa, bu limit $f(x)$ funksiyaning $[a,b)$ bo‘yicha xosmas integrali deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi:

$$\int_a^b f(x)dx=\lim_{t\to b-0}F(t)=\lim_{t\to b-0}\int_a^b f(x)dx \qquad (3)$$

Ta’rif. Agar $t\to b-0$ da $F(t)$ funksiyaning limiti mavjud va chekli bo‘lsa, (3) xosmas integral yaqinlashuvchi deyiladi. $f(x)$ esa $[a,b)$ da integrallanuvchi funksiya deyiladi.

Agar $t\to b-0$ da $F(t)$ funksiyaning limiti cheksiz bo‘lsa yoki limit mavjud bo‘lmasa, (3) xosmas integral uzoqlashuvchi deyiladi.

17.7. Yaqinlashuvchi xosmas integralning xossalari

$f(x)$ va $g(x)$ funksiyalar $[a,b)$ da berilgan bo‘lib, b nuqta shu funksiyalarning maxsus nuqtasi bo‘lsin.

1^0. Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ xosmas integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^b[\alpha f(x)\pm\beta g(x)]dx$$

xosmas integral ham yaqinlashuvchi bo‘lib,

$$\int_a^b[\alpha f(x)\pm\beta g(x)]dx=\alpha\int_a^b f(x)dx\pm\beta\int_a^b g(x)dx$$

bo‘ladi, bu yerda α,β -o‘zgarmas sonlar.

2^0. Agar ixtiyoriy $x \in [a,b)$ uchun $f(x) \le g(x)$ bo‘lib, $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo‘ladi.

3^0. *Nyuton- Leybnis formulasi*. $f(x)$ funksiya $[a,b)$ oraliqda uzluksiz bo‘lib, $F(x)$ esa uning shu oraliqdagi boshlang‘ich funksiyasi bo‘lsin ($F'(x) = f(x)$).

Unda

$$\int_a^b f(x)dx = F(x)\Big|_a^{b-0} = F(b-0) - F(a) \qquad (4)$$

bo‘ladi. Bu yerda

$$F(b-0) = \lim_{t \to b-0} F(t).$$

4^0. *O‘zgaruvchini almashtirish formulasi.* $f(x)$ funksiya $[a,b)$ da uzluksiz, $\varphi(t)$ funksiya esa $[\alpha,\beta)$ da uzluksiz differensiallanuvchi funksiya bo‘lib,

$$a = \varphi(a) \le \varphi(t) < \lim_{t \to b-0} \varphi(t) = b$$

bo‘lsa, u holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(x))\varphi'(t)dt$$

bo‘ladi.

5^0. *Bo‘laklab integrallash formulasi*. Agar $u = u(x)$ va $v = v(x)$ funksiyalar $[a,b)$ da uzluksiz differensiallanuvchi bo‘lib, $\lim_{t \to b-0}(uv)$ mavjud bo‘lsa, u holda

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

bo‘ladi. Bu yerda

$$uv\Big|_a^b = \lim_{t \to b-0} u(t)v(t) - u(a)v(a)$$

Misol. Ushbu

$$\int_1^2 \frac{dx}{x\sqrt{\ln x}}$$

integralni hisoblang. Ravshanki,

$$f(x)=\frac{1}{x\sqrt{\ln x}}$$

funksiyaning (1,2] oraliqdagi boshlang‘ich funksiyasi

$$F(x)=2\cdot\sqrt{\ln x}$$

bo‘ladi. Nyuton-Leybnits formulasidan foydalanib, topamiz:

$$\int_1^2\frac{dx}{x\sqrt{\ln x}}=(2\sqrt{\ln x})\Big|_{1-0}^{2}=2\ln 2-2\lim_{t\to 1-0}\sqrt{\ln t}=2\ln 2$$

17.8. Xosmas integralning yaqinlashuvchiligi haqida teoremalar

$f(x)$ funksiya $[a,b)$ da berilgan bo‘lib, b nuqta shu funksiyaning maxsus nuqtasi bo‘lsin.

Teorema. $[a,b)$ da manfiy bo‘lmagan $f(x)$ funksiyaning $\int_a^b f(x)dx$ xosmas integralning yaqinlashuvchi bo‘lishi uchun $\forall t\in(a,b)$ da

$$F(t)=\int_a^b f(x)dx\le C \qquad (C=const)$$

bo‘lishi zarur va yetarli.

Teorema. $f(x)$ va $g(x)$ funksiyalar $[a,b)$ da berilgan bo‘lib, b nuqta shu funksiyalarning maxsus nuqtasi bo‘lsin. Agar $\forall x\in[a,b)$ da

$$0\le f(x)\le g(x)$$

bo‘lsa, u holda $\int_a^b g(x)dx$ integralning yaqinlashuvchiligidan $\int_a^b f(x)dx$ ning yaqinlashuvchiligi; $\int_a^b f(x)dx$ integralning uzoqlashuvchiligidan $\int_a^b g(x)dx$ ning uzoqlashuvchiligi kelib chiqadi.

Misol. Ushbu

$$\int_0^1\frac{\cos^2 x}{\sqrt[4]{1-x}}dx$$

integralning yaqinlashuvchiligini ko‘rsating. Ravshanki, $\forall x\in[0,1)$ da

$$\frac{\cos^2 x}{\sqrt[4]{1-x}}\le\frac{1}{\sqrt[4]{1-x}}$$

bo‘ladi. Ushbu

$$\int_0^1\frac{1}{\sqrt[4]{1-x}}dx=\int_0^1\frac{dx}{(1-x)^{1/4}}$$

xosmas integralning yaqinlashuvchiligini e’tiborga olib, yuqoridagi teoremadan foydalanib,

$$\int_0^1 \frac{\cos^2 x}{\sqrt[4]{1-x}}dx$$

integralning yaqinlashuvchi ekanini topamiz.

Teorema. Agar x ning b ga yetarli yaqin qiymatlarida

$$f(x)=\frac{\varphi(x)}{(b-x)^{\alpha}}\,(\alpha>0)$$

bo‘lsa, u holda $\varphi(x)\le C<+\infty$ va $\alpha<1$ bo‘lganda $\int_a^b f(x)dx$ integral yaqinlashuvchi, $\varphi(x)\ge C>0$ va $\alpha\ge 1$ bo‘lganda $\int_a^b f(x)dx$ integral uzoqlashuvchi bo‘ladi.

Teorema. Agar $x\to b-0$ da $f(x)$ funksiya $\frac{1}{b-x}$ ga nisbatan $\alpha(\alpha>0)$ tartibli cheksiz katta bo‘lsa, u holda $\int_a^b f(x)dx$ integral $\alpha<1(\alpha\ge 1)$ bo‘lganda yaqinlashuvchi (uzoqlashuvchi) bo‘ladi.

Teorema (*Koshi teoremasi*). Quyidagi

$$\int_a^b f(x)dx$$

xosmas integralning (b-maxsus nuqta) yaqinlashuvchi bo‘lishi uchun, ixtiyoriy $\varepsilon>0$ son olinganda ham, shunday $\delta>0$ topilib, $b-\delta<t'<b, b-\delta<t''<b$ tengsizliklarni qanoatlantiruvchi ixtiyoriy t' va t'' lar uchun

$$|F(t'')-F(t')|=\left|\int_a^{t''} f(x)dx-\int_a^{t'} f(x)dx\right|=\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Teorema (*Dirixle alomati*). $f(x)$ va $g(x)$ funksiyalar $[a,b)$ da berilgan bo‘lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,b)$ da uzluksiz va uning shu oraliqdagi boshlang‘ich $F(x)$ ($F'(x)=f(x)$) funksiyasi chegaralangan,

2) $g(x)$ funksiya $[a,b)$ da $g'(x)$ hosilaga ega va uzluksiz funksiya,

3) $g(x)$ funksiya $[a,b)$ da kamayuvchi,

4) $\lim\limits_{x\to b-0} g(x)=0.$

U holda

$$\int_a^b f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

17.9. Xosmas integralning absolyut yaqinlashuvchiligi

Ta'rif. Agar $\int_a^b |f(x)|dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^b f(x)dx$ absolyut yaqinlashuvchi integral, $f(x)$ funksiya esa $[a,b)$ da absolyut integrallanuvchi funksiya deb ataladi.

Xosmas integrallar mavzusiga doir misollar

1. Quyidagi xosmas integralni hisoblang.

1) $\int_1^\infty \frac{dx}{x^4}$; 2) $\int_1^\infty \frac{dx}{\sqrt{x}}$

3) $\int_0^\infty e^{-2x}dx$; 4) $\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2}$;

5) $\int_2^\infty \frac{\ln x}{x}dx$; 6) $\int_1^\infty \frac{dx}{x^2(x+1)}$,

7) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$; 8) $\int_0^{+\infty} x\sin x dx$;

9) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$; 10) $\int_1^2 \frac{xdx}{\sqrt{x-1}}$;

11) $\int_a^b \frac{xdx}{(x-a)(x-b)}$; 12) $\int_0^1 \frac{dx}{x^3-5x^2}$.

2. Quyidagi integrallarni yaqinlashuvchanlikka tekshiring.

1) $\int_a^{+\infty} \frac{dx}{x^\alpha}$, $(a>0,\ \alpha>0)$; 2) $\int_1^{+\infty} \frac{dx}{x+x^2}$;

3) $\int_0^\infty \frac{x}{x^3+1}dx$; 4) $\int_1^\infty \frac{x^3+1}{x^4}dx$;

5) $\int_1^\infty \frac{dx}{x\sqrt[3]{x^2+1}}$; 6) $\int_0^2 \frac{dx}{\ln x}$;

7) $\int_1^\infty \frac{\sqrt{x}}{\sqrt{1-x^4}}dx$ is 8) $\int_0^1 \frac{\sqrt{x}dx}{e^{\sin x}-1}$

3. Quyidagi xosmas integrallarni uzoqlashuvchilikka tekshiring.

1) $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$; 2) $\int_1^{+\infty} \frac{dx}{x\ln x}$;

3) $\int_{-\infty}^0 \frac{dx}{1+x}$; 4) $\int_0^{+\infty} \sin x dx$;

5) $\int_0^{+\infty} \frac{xdx}{x^2-1}$; 6) $\int_1^{+\infty} \frac{\ln(x^2+x)}{x}dx$;

7) $\int\limits_{1}^{+\infty}\frac{dx}{\sqrt{4+x^2}}$; 8) $\int\limits_{3}^{+\infty}\frac{2x+5}{x^2+3x-10}dx$.

4. Quyidagi integrallarni shartli yaqinlashuvchiligini ko'rsating.

1) $\int\limits_{1}^{+\infty}\frac{\sin x}{x}dx$; 2) $\int\limits_{0}^{+\infty}\sin x^2dx$;

3) $\int\limits_{0}^{+\infty}\frac{\sin \ln x}{\sqrt{x}}dx$; 4) $\int\limits_{0}^{+\infty}\frac{\cos x}{\sqrt{x+\ln x}}dx$;

5) $\int\limits_{2}^{+\infty}\frac{\sqrt{x+1}\sin x}{\ln x}dx$; 6) $\int\limits_{0}^{+\infty}\frac{\cos x}{\sqrt{x+\ln x}}dx$;

7) $\int\limits_{0}^{+\infty}\frac{\sin \sin x}{\sqrt{x}}dx$; 8) $\int\limits_{1}^{+\infty}\frac{1}{\sqrt{x}}\sin\left(x+\frac{1}{x}\right)dx$.

5. Quyidagi integrallarni absolyut yaqinlashuvchiligini isbotlang.

1) $\int\limits_{1}^{+\infty}\frac{\sin x}{1+x^2}dx$; 2) $\int\limits_{1}^{+\infty}\frac{\sin x}{x^2}dx$; 3) $\int\limits_{1}^{+\infty}\ln^2\left(1+\frac{1}{x}\right)\sin xdx$;

4) $\int\limits_{1}^{+\infty}\frac{1+x}{x^3}\sin x^3dx$; 5) $\int\limits_{1}^{+\infty}\frac{\sin(x+x^2)}{x\sqrt{x}}dx$.

18-§. PARAMETRGA BOG'LIQ INTEGRALLAR

18.1. Parametrga bog'liq integral tushunchasi

$f(x,y)$ funksiya

$$D=\{(x,y)\in R^2 : a\le x\le b,\ y\in E\subset R\}$$

to'plamda berilgan bo'lsin. y o'zgaruvchining har bir tayin qiymatida $f(x,y)$ funksiya x o'zgaruvchisi bo'yicha $[a,b]$ da integrallanuvchi, ya'ni

$$\int\limits_a^b f(x,y)dx$$

integral mavjud bo'lsin.

Bu integral y o'zgaruvchining E dan olingan qiymatiga bog'liq bo'ladi:

$$I(y)=\int\limits_a^b f(x,y)dx \qquad (1)$$

Odatda (1) perametrga bog'liq integral, y o'zgaruvchi esa parametr deyiladi.

Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham, (ixtiyoriy $\forall x \in [a,b]$) $\delta = \delta(\varepsilon, x) > 0$ topilsaki, $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $y \in E$ uchun

$$|f(x,y) - \varphi(x)| < \varepsilon, \quad x \in [a,b]$$

bo'lsa, u holda $\varphi(x)$ funksiya $f(x,y)$ ning $y \to y_0$ dagi ***limit funksiyasi*** deyiladi.

$$\left(\lim_{y \to y_0} f(x,y) = \varphi(x) \right).$$

Ta'rif. Agar ixtiyoriy $\varepsilon > 0$ olinganda ham shunday $\Delta = \Delta(x,\varepsilon) > 0$ topilsaki, $|y| > \Delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $y \in E$ uchun

$$|f(x,y) - \varphi(x)| < \varepsilon$$

bo'lsa, u holda $\varphi(x)$ funksiya $f(x,y)$ ning $y \to \infty$ dagi ***limit funksiyasi*** deyiladi.

Teorema. $f(x,y)$ funksiya $y \to y_0$ da limit funksiya $\varphi(x)$ ga ega bo'lib, unga tekis yaqinlashishi uchun ixtiyoriy $\varepsilon > 0$ olinganda ham, x $(x \in [a,b])$ ga bog'liq bo'lmagan shunday $\delta = \delta(\varepsilon) > 0$ topilib, $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy $y, y' \in E$ hamda ixtiyoriy $x \in [a,b]$ uchun

$$|f(x,y) - f(x,y')| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarlidir.

Misol. Ushbu

$$f(x,y) = \sin\frac{x}{y}$$

funksiya $D = \{(x,y) \in R^2 : x \in R, 0 < y < +\infty\}$ to'plamda berilgan bo'lsin. $y \to +\infty$ da limit funksiyani toping va intilish xarakterini tekshiring.

$$\lim_{y \to +\infty} f(x,y) = \lim_{y \to +\infty} \sin\frac{x}{y} = 0$$

ekanini ko'rish qiyin emas: $\varphi(x) = 0$. Demak,

$$|f(x,y) - \varphi(x)| = \left|\sin\frac{x}{y}\right| \le \left|\frac{x}{y}\right| < \varepsilon \Rightarrow y > \frac{|x|}{\varepsilon}.$$

Ixtiyoriy $\varepsilon > 0$ ga ko'ra $\Delta = \frac{|x|}{\varepsilon}$ desak, u holda $|y| > \Delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy y uchun $\left|\sin\frac{x}{y}\right| < \varepsilon$ bo'ladi. Bu yerda $\Delta = \frac{|x|}{\varepsilon}$ faqatgina ε ga bog'liq bo'lmay x ga ham bog'liqdir. Δ ni x ga bog'liqmas qilib olib bo'lmasligini ko'rsatishni o'quvchiga havola etamiz. Demak, qaralayotgan funksiya o‘z limit funksiyasiga notekis yaqinlashadi.

18.2. Parametrga bog‘liq integrallarning funksional xossalari

Teorema. $f(x,y)$ funksiya y ning E to‘plamdan olingan har bir tayin qiymatida x ning funksiyasi sifatida $[a,b]$ oraliqda uzluksiz bo‘lsin. Agar $f(x,y)$ funksiya $y \to y_0$ da $\varphi(x)$ limit funksiyaga ega bo‘lsa va unga tekis yaqinlashsa, u holda

$$\lim_{y\to y_0}\int_a^b f(x,y)dx = \int_a^b \varphi(x)dx \qquad (2)$$

bo‘ladi.

Teorema. Agar $f(x,y)$ funksiya $D = \{(x,y) \in R^2 : x \in [a,b], y \in [c,d]\}$ to‘plamda uzluksiz bo‘lsa, u holda

$$I(y) = \int_a^b f(x,y)dx$$

funksiya $[c,d]$ oraliqda uzluksiz bo‘ladi.

Teorema. $f(x,y)$ funksiya D to‘plamda berilgan va y o‘zgaruvchining $[c,d]$ oraliqdan olingan har bir tayin qiymatida x o‘zgaruvchining funksiyasi sifatida $[a,b]$ oraliqda uzluksiz bo‘lsin. Agar $f(x,y)$ funksiya D da $f_y'(x,y)$ xususiy hosilaga ega bo‘lib, y D da uzluksiz bo‘lsa, u holda $I(y)$ funksiya ham $[c,d]$ oraliqda $I'(y)$ hosilaga ega va ushbu

$$I'(y) = \int_a^b f_y'(x,y)dx \qquad (3)$$

munosabat o‘rinlidir.

Teorema. Agar $f(x,y)$ funksiya $D = \{(x,y) \in R^2 : x \in [a,b], y \in [c,d]\}$ to‘plamda uzluksiz bo‘lsa, u holda $\int_c^d I(y)dy$ integral mavjud va

$$\int_c^d \left[\int_a^b f(x,y)dx\right]dy = \int_a^b \left[\int_c^d f(x,y)dy\right]dx \qquad (4)$$

munosabat o‘rinlidir.

Teorema. $f(x,y)$ funksiya D to‘plamda uzluksiz, $\alpha(y), \beta(y)$ funksiyalar $[c,d]$ da uzluksiz va

$$a \le \alpha(y) \le \beta(y) \le b \qquad (5)$$

tengsizlikni qanoatlantirsin. U holda

$$\tilde{I}(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya ham $[c,d]$ oraliqda uzluksiz bo‘ladi.

18.3. Leybnis formulasi

Teorema. $f(x,y)$ funksiya D to‘plamda uzluksiz, $f'_y(x,y)$ xususiy hosilaga ega va D da uzluksiz, $\alpha(y),\beta(y)$ funksiyalar $\alpha'(y),\beta'(y)$ hosilalarga ega va ular (5) shartni qanoatlantirsin. U holda $I(y)$ funksiya ham $[c,d]$ oraliqda hosilaga ega va

$$I(y)=\int\limits_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx+\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y) \qquad (6)$$

munosabat o‘rinlidir. (6) ga ***Leybnis formulasi*** deyiladi.

18.4. Parametrga bog‘liq xosmas integrallar

$f(x,y)$ funksiya $D=\{(x,y)\in R^2 : x\in[a,+\infty), y\in E\subset R\}$ to‘plamda berilgan bo‘lib,

$$\int\limits_a^{+\infty} f(x,y)dx\ , \quad (y\in E)$$

xosmas integral mavjud va chekli bo‘lsin. Bu integral y ning qiymatiga bog‘liq bo‘lib,

$$I(y)=\int\limits_a^{+\infty} f(x,y)dx$$

integral parametrga bog‘liq (chegarasi cheksiz) ***xosmas integral*** deb ataladi.

Chegarasi cheksiz bo‘lgan xosmas integral ta’rifiga ko‘ra ixtiyoriy $[a,A]$ da $(a<A<+\infty)$

$$I(A,y)=\int\limits_a^{A} f(x,y)dx \qquad (7)$$

integral mavjud va

$$I(y)=\int\limits_a^{+\infty} f(x,y)dx=\lim_{A\to+\infty} I(A,y)\,. \qquad (8)$$

Demak, $I(y)$ va $I(A,y)$ funksiyalar (8) va (7) integrallar orqali aniqlangan bo‘lib, $I(y)$ $I(A,y)$ funksiyaning $A\to+\infty$ dagi limit funksiyasidir.

Ta’rif. Agar $A\to+\infty$ da $I(A,y)$ funksiya o‘z limit funksiyasi $I(y)$ ga E to‘plamda tekis (notekis) yaqinlashsa, u holda

$$I(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to‘plamda tekis (notekis) yaqinlashuvchi deb aytiladi.

Teorema (***Koshi teoremasi***). (8) integral E to‘plamda tekis yaqinlashuvchi bo‘lishi uchun ixtiyoriy $\varepsilon>0$ olinganda ham, shunday $\Delta=\Delta(\varepsilon)>0$ topilsaki, $A'>\Delta, A''>\Delta$ tengsizliklarni qanoatlantiruvchi A', A'' va ixtiyoriy $y\in E$ uchun

$$\left|\int_{A'}^{A''} f(x,y)dx\right|<\varepsilon$$

tengsizlikning bajarilishi zarur va yetarlidir.

Tekis yaqinlashishga tekshirish uchun Veyershtrass alomati: $f(x,y)$ funksiya D to‘plamda berilgan va

$$I(y)=\int_a^{+\infty} f(x,y)dx$$

integral mavjud bo‘lsin.

Agar shunday $\varphi(x)$ funksiya topilib $(x\in[a,+\infty))$,

1) ixtiyoriy $x\in[a,+\infty)$ va ixtiyoriy $y\in E$ uchun $f(x,y)\le\varphi(x)$ bo‘lsa,

2) $\int_a^{+\infty}\varphi(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u holda

$$I(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to‘plamda tekis yaqinlashuvchi bo‘ladi.

Misol. Ushbu

$$\int_0^{+\infty}\frac{arctgxy}{1+x^2}dx,\quad y\in R$$

integralni tekis yaqinlashishga tekshiring.

Agar

$$\left|\frac{arctgxy}{1+x^2}\right|\le\frac{\pi}{2(1+x^2)}$$

ekanini hisobga olsak va $\varphi(x)=\dfrac{\pi}{2(1+x^2)}$ deb olinsa, u holda

$$\frac{\pi}{2}\int_0^{+\infty}\frac{dx}{1+x^2}=\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}$$

bo‘lgani uchun Veyershtrass alomatiga ko‘ra berilgan integral R da tekis yaqinlashuvchi bo‘ladi.

18.5. Parametrga bog‘liq xosmas integrallarning funksional xossalari

$f(x,y)$ funksiya D to‘plamda berilgan va y_0 E to‘plamning limit nuqtasi bo‘lsin.

Teorema. $f(x,y)$ funksiya

1) y o‘zgaruvchining E dan olingan har bir tayin qiymatida x o‘zgaruvchining funksiyasi sifatida $[a,+\infty)$ da uzluksiz,

2) $y \to y_0$ da ixtiyoriy $[a,A]$ $(a < A < +\infty)$ oraliqda $\varphi(x)$ limit funksiyaga tekis yaqinlashuvchi bo‘lsin.

Agar

$$I(y) = \int_a^{+\infty} f(x,y)dx$$

integral E to‘plamda tekis yaqinlashuvchi bo‘lsa, u holda $y \to y_0$ da $I(y)$ funksiya limitga ega va

$$\lim_{y \to y_0} \int_a^{+\infty} f(x,y)dx = \int_a^{+\infty} \varphi(x)dx$$

munosabat o‘rinli bo‘ladi.

Teorema. $f(x,y)$ funksiya D to‘plamda uzluksiz va

$$I(y) = \int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ oraliqda tekis yaqinlashuvchi bo‘lsa, u holda $I(y)$ funksiya $[c,d]$ da uzluksiz bo‘ladi.

Teorema. $f(x,y)$ funksiya D to‘plamda uzluksiz, $f'_y(x,y)$ xususiy hosilaga ega va D da uzluksiz bo‘lib, $y \in [c,d]$ da

$$I(y) = \int_a^{+\infty} f(x,y)dx$$

integral yaqinlashuvchi bo‘lsin.

Agar $\int_a^{+\infty} f'_y(x,y)dx$ integral $[c,d]$ da tekis yaqinlashuvchi bo‘lsa, u holda $I(y)$ funksiya ham $[c,d]$ oraliqda $I'(y)$ hosilaga ega va

$$I'(y) = \int_a^{+\infty} f'_y(x,y)dx$$

munosabat o‘rinli bo‘ladi.

Teorema. $f(x,y)$ funksiya D to‘plamda uzluksiz va

$$I(y) = \int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ oraliqda tekis yaqinlashuvchi bo‘lsin. U holda $I(y)$ funksiya $[c,d]$ oraliqda integrallanuvchi va

$$\int_c^d I(y)dy=\int_c^d\left[\int_a^{+\infty} f(x,y)dx\right]dy=\int_a^{+\infty}\left[\int_c^d f(x,y)dy\right]dx$$

munosabat o‘rinli.

$f(x,y)$ funksiya $D=\{(x,y)\in R^2 : x\in[a,+\infty), y\in[c,+\infty)\}$ to‘plamda berilgan bo‘lsin.

Misol. Ushbu

$$\int_0^1 x^{p-1}\ln^q\frac{1}{x}dx,\ p\geq p_0>0$$

integralni tekis yaqinlashishga tekshiring.

Ushbu $x=e^{-t}$ $(t<0)$ almashtirish natijasida integral $\int_0^{+\infty} t^q e^{-pt}dt$ ko‘rinishga keladi.

$$\left|t^q e^{-pt}\right|\leq\frac{t^q}{e^{p_0 t}}$$

bo‘lib, $\int_0^{+\infty}\frac{t^q}{e^{p_0 t}}dt$ integralga yaqinlashuvchi ekanini ko‘rish mumkin. Demak, Veyershtrass alomatiga ko‘ra, berilgan integral tekis yaqinlashuvchi.

18.6. Eyler integrallari

1. *Beta funksiya (I tur Eyler integrali*).

Ushbu $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ $(a>0, b>0)$ integral ***beta funksiya*** yoki I tur Eyler integrali deb ataladi.

Beta funksiyaning xossalari:

1. $B(a,b)=B(b,a)$.

2. $B(a,b)=\frac{b-1}{a+b-1}B(a,b-1)$ $(b>1, a>0)$.

2`. $B(a,n)=\frac{n-1}{a+n-1}\cdot\frac{n-2}{a+n-2}\cdot\ldots\cdot\frac{1}{a+1}B(a,1),\quad n\in N$.

3. $B(a,1-a)=\frac{\pi}{\sin a\pi}$ $(0<a<1)$.

4. $B(\frac{1}{2},\frac{1}{2})=\pi$.

2. *Gamma funksiya* (*II tur Eyler integrali*).

Ushbu $\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx$ $(a>0)$ integral ***gamma funksiya*** yoki II tur Eyler integrali deb ataladi.

Gamma funksiya xossalari:

1. $\Gamma(a)=\lim\limits_{n\to\infty}n^a\dfrac{1\cdot 2\cdot 3\cdot ...\cdot(n-1)}{a(a+1)...(a+n-1)}$

2. $\Gamma(a+1)=a\Gamma(a)$

2`. $\Gamma(n+1)=n!$

2``. $\Gamma(1)=\Gamma(2)=1$.

3. $\Gamma(a)$ $(0,+\infty)$ da uzluksiz va barcha tartibdagi uzluksiz hosilalarga ega va

$$\Gamma^{(n)}(a)=\int_0^{+\infty}x^{a-1}e^{-x}(\ln x)^n dx \qquad (n=1,2,\ldots).$$

4. $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

5. $\Gamma(a)\Gamma(1-a)=B(a,1-a)=\dfrac{\pi}{\sin a\pi}$.

Xususan, $a=\frac{1}{2}$ da

$$\Gamma(\tfrac{1}{2})=\sqrt{\pi}\ .$$

6. $\Gamma(a)\Gamma(a+\frac{1}{2})=\dfrac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a)$ $(0<a<1)$ **(*Lejandr formulasi*).**

Misol. Ushbu

$$I=\int_0^{+\infty}e^{-x^2}dx$$

integralni hisoblang.

$x^2=t$ almashtirish natijasida integral quyidagi ko'rinishga keladi:

$$I=\frac{1}{2}\int_0^{+\infty}\frac{e^{-t}}{\sqrt{t}}dt=\frac{1}{2}\int_0^{+\infty}e^{-t}\cdot t^{-\frac{1}{2}}dt=\frac{1}{2}\int_0^{+\infty}t^{\frac{1}{2}-1}e^{-t}dt=\frac{1}{2}\Gamma(\frac{1}{2})$$

Yuqoridagi (5) munosabatdan foydalanib $I=\frac{1}{2}\sqrt{\pi}$ ekanini topamiz. Demak,

$$\int_0^{+\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

Misol. Ushbu

$$I=\int_0^{\frac{\pi}{2}}\sin^6 x\cos^4 x dx$$

integralni hisoblang.

$\sin x = \sqrt{t} \quad (t > 0)$ almashtirish natijasida integral quyidagi ko‘rinishga keladi:

$$\int_0^{\frac{\pi}{2}} (\sin^2 x)^3 (1-\sin^2 x)^2 dx = \frac{1}{2}\int_0^1 (1-t)^{\frac{3}{2}} \cdot t^{\frac{5}{2}} dt =$$

$$= \frac{1}{2}\int_0^1 (1-t)^{\frac{5}{2}-1} \cdot t^{\frac{7}{2}-1} dt = \frac{1}{2} B\left(\frac{5}{2}, \frac{7}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)} =$$

$$= \frac{1}{2}\cdot\frac{3}{4}\sqrt{\pi}\cdot\frac{15}{8}\cdot\sqrt{\pi}\cdot\frac{1}{120} = \frac{3\pi}{512}$$

M13. Parametrga bog‘liq integrallarga doir mashqlar

1. Quyidagi funksiyalarni berilgan to’plamda limit funksiyasini toping.

1) $f(x,y) = x^4 cos\frac{1}{xy}$, $D = \{(x,y) \in R^2 : 1 \le x < +\infty, 0 < y < +\infty\}$, $y_o = +\infty$

2) $f(x,y) = \sqrt{y} sin\frac{x}{y\sqrt{y}}$, $D = \{(x,y) \in R^2 : x \in R^2, 0 < y < +\infty\}$, $y_o = +\infty$

3) $f(x,n) = x^{2n}$, $D = \{(x,n) \in R^2 : 1 \le x < \delta, 0 < \delta < 1,\ n \in N\}$, $n_o = \infty$

4) $f(x,y) = \frac{nx}{1+n^3x^2}$, $D = \{(x,n) \in R^2 : 1 \le x < +\infty,\ n \in N\}$, $n_o = \infty$

5) $f(x,y) = \frac{n^2x^2}{1+n^2x^4} sin\frac{x^2}{\sqrt{n}}$, $D = \{(x,n) \in R^2 : 1 \le y < +\infty,\ n \in N\}$, $n_o = \infty$

2. Quyidagi funksiyalarning berilgan to’plamda limit funksiyasini toping va uni tekis yaqinlashishga tekshiring.

1) $f(x,y) = e^{-yx^2}$, $D = \{(x,y) \in R^2 : 1 \le x < +\infty, 0 < y < +\infty\}$, $y_o = +\infty$

2) $f(x,y) = \sqrt{y} sin\frac{x}{y\sqrt{y}}$, $D = \{(x,y) \in R^2 : x \in R^2, 0 < y < +\infty\}$, $y_o = +\infty$

3) $f(x,n) = x^{2n}$, $D = \{(x,n) \in R^2 : 1 \le x < \delta, 0 < \delta < 1,\ n \in N\}$, $n_o = \infty$

4) $f(x,y) = \frac{nx}{1+n^3x^2}$, $D = \{(x,n) \in R^2 : 1 \le x < +\infty,\ n \in N\}$, $n_o = \infty$

3. Quyidagi integrallarni hisoblang.

1) $\lim\limits_{\propto\to 0} \int_{-1}^{1} \sqrt{x^2 + \propto^2}\, dx$ 2) $\lim\limits_{\propto\to 0} \int_0^{1+\propto} \frac{dx}{1+x^2+\propto^2}$

3) $\lim\limits_{\propto\to 0} \int_0^1 \frac{dx}{1+\left(1+\frac{x}{n}\right)^n}$ 4) $\lim\limits_{\propto\to 0} \int_0^y \frac{\ln(1+|\propto|)}{\ln(x^2+\propto^2)} dx$

5) $\lim\limits_{\propto\to 0} \int_{\propto}^{1+\propto} \frac{dx}{1+x^2+\propto^2}$ 6) $\lim\limits_{\propto\to 0} \int_{-1}^{1} \frac{dx}{1+x^2+\propto^2}$

4. Quyidagi funksiyalarning hosilalarini toping

1) $F(x,y) = \int_{x+1}^{x^2+1} e^{-xy^2} dy$ 2) $F(x,\propto) = \int_{cos\propto}^{sin\propto} e^{\propto\sqrt{1-x^2}} dx$

3) $F(\propto) = \int_{\propto_1}^{\propto_2} f(x+\propto,\ x-\propto)\, dx$ 4) $F(\propto) = \int_{\propto^2+3}^{\propto^3} \frac{\ln(1+\propto x)}{x} dx$

3) $F(x,\propto) = \int_{cos\propto}^{sin\propto} \frac{\ln(1+\propto x)}{x} dx$

5. Quyidagi integrallarni tekis yaqinlashishga tekshiring.

1) $\int_1^{+\infty} \frac{ln^p x}{x\sqrt{x}} dx$, $(0 \le p \le 10)$

2) $\int_0^{+\infty} e^{-\propto x} sinx\, dx$, $(0 < \propto_0 \le \propto + \infty)$

3) $\int_0^{+\infty} \frac{arctgxy\cdot arctgxy^2x}{1+x^2} e^{-xy} dx,\ y \in [0,+\infty)$;

4) $\int_0^{+\infty} xe^{-x^2} sinxydx,\ y \in R$

5) $\int_1^{+\infty} x^{\propto} e^{-x}\, dx,\ (a \leq \propto \leq b)$

6) $\int_{-\infty}^{+\infty} \frac{cos\propto x}{1+x^2} dx,\ \propto \in R$

7) $\int_0^{+\infty} \frac{sinax - sinbx}{x} dx,\ (a > 0,\ b > 0)$

8) $\int_0^{+\infty} \left(\frac{e^{-\propto x} - e^{-\beta x}}{x}\right)^2 dx,\ (\propto > 0,\ \beta > 0)$

9) $\int_0^{+\infty} \frac{arctg\propto x}{x^2\sqrt{x^2-1}} dx$

10) $\int_0^{+\infty} \frac{\ln(\propto^2 + x^2)}{\beta^2 + x^2} dx$

6. Eyler integrallaridan foydalanib quyidagi integrallarni hisoblang.

1) $I = \int_0^{+\infty} e^{-x^2} dx$

2) $I = \int_0^{\frac{\pi}{2}} sin^2xcos^4x\, dx$

3) $I = \int_0^{+\infty} x^{2n} e^{-x^2}\, dx,\ (n \in N)$

4) $\int_0^1 \sqrt{x - x^2}\, dx$

5) $\int_0^0 x^2\sqrt{a^2 - x^2}\, dx,\ (a > 0)$

6) $\int_0^{+\infty} \frac{x^2}{1+x^4} dx$

7) $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} dx,\ (n > 1)$

8) $\int_0^{+\infty} \frac{sin^2x}{x^2} dx$

7. Testlar

1) Parametrga bog'liq integrallarni aniqlang.

1) $\int_0^A (x^2 + y^2)\, dx$ 2) $\int_0^1 (xy)\, dx$ 3) $\int_0^1 (x^2 + y^2)\, dy$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

2) $f(x,y) = \frac{xy}{x+y}$ bo'lsa qaysi integral $[0,\ 2]$ kesmada uzluksiz bo'ladi.

1) $\int_0^1 f(x,y)\, dx$ 2) $\int_{-1}^1 f(x,y)\, dx$ 3) $\int_2^0 f(x,y)\, dx$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

3) $f(x,y) = \frac{xy}{x+y}$ bo'lsa, qaysi integral $[1,2]$ kesmada differensiallanuvchi bo'ladi.

1) $\int_0^1 f(x,y)\, dx$ 2) $\int_{-1}^1 f(x,y)\, dx$ 3) $\int_2^0 f(x,y)\, dx$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

4) $(0,+\infty)$ oraliqda tekis yaqinlashuvchi integralni aniqlang.

1) $\int_0^{+\infty} e^{-x}\sin(xy)\, dx$ 2) $\int_0^{+\infty} \frac{sinxy}{1+x^2} dx$ 3) $\int_0^{+\infty} e^{-yx^2}\, dx$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

5) To'g'ri tenglikni toping.

1) $\int_0^{+\infty} \frac{sinx}{x} dx = \frac{\pi}{2}$ 2) $\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$ 3) $\int_0^{+\infty} cosx^2\, dx = \frac{\sqrt{\pi}}{2}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

6) To'g'ri tenglikni aniqlangan.

1) $B(x,y)=B(y,x)$ 2) $B(x,\ (-x))=F(x)\cdot F(1-x)$

3) $\Gamma(x)\cdot\Gamma(y)=B(x,y)\cdot\Gamma(x+y)$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

7) Beta funksiyaning nechta maxsus nuqtasi bor.

A) 1 B) 2 C) 3 D) 4 E) 5

8) To'g'ri tenglikni aniqlang.

1) $\Gamma(n+1)=n!$ 2) $B(a,b)=\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$ 3) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

9)To'g'ri tenglikni aniqlang.

1) $B(a,1-a)=\frac{\pi}{sina\pi}\ (0<a<1)$ 2) $B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$

3) $B(a,b)=\frac{b-1}{a+b-1}B(a,b-1)$, (b>1)

A) 1 B) 2 C) 3 D) 1, 2 E) 1, 3

19-§. KARRALI INTEGRALLAR

19.1. Ikki karrali integral ta'riflari

$f(x,y)$ funksiya $(D)\subset R^2$ sohada berilgan bo'lsin. Bu sohaning $P\in$ T bo'linishi va bu bo'linishlarning har bir kvadratlanuvchi (yuzaga ega) (D_k) $(k=1,2,\ldots,n)$ bo'lagida ixtiyoriy (ξ_k,η_k) nuqta olib,

$$\sigma=\sum_{k=1}^{n}f(\xi_k,\eta_k)D_k \tag{1}$$

yig'indini tuzaylik, bunda $D_k-(D_k)$ sohaning yuzi.

Odatda (1) $f(x,y)$ funksiyaning integral yig'indisi yoki ***Riman yig'indisi*** deb ataladi.

Ta'rif. Ixtiyoriy $\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilsaki, (D) sohaning diametri $\lambda_p<\delta$ bo'lgan har qanday bo'linishi hamda har bir (D_k) bo'lakdagi ixtiyoriy (ξ_k,η_k) nuqtalar uchun

$$|\sigma-I|<\varepsilon$$

tengsizlik bajarilsa, u holda funksiya integrallanuvchi va I songa $f(x,y)$ funksiyaning (D) soha bo'yicha ikki karrali integrali (Riman itegrali) deyiladi va

$$\iint\limits_{(D)}f(x,y)dD \qquad \left(\iint\limits_{(D)}f(x,y)dxdy\right)$$

kabi belgilanadi.

Misol. Ushbu

$$\iint\limits_{(D)} xydD, \qquad (D)=\{(x,y)\in R^2 : 0\le x\le 1, 0\le y\le 1\}$$

integralni yuqoridagi ta’rif yordamida hisoblang.

Ravshanki, $f(x,y)=xy$ funksiya (D) da uzluksiz, demak u (D) da integrallanuvchi bo‘ladi. (D) sohani $x=\frac{i}{n}$, $y=\frac{j}{n}$ $\left(i,j=\overline{1,n-1}\right)$ chiziqlar yordamida bo‘laklarga ajratamiz va har bir (D_{ij}) da $(\xi_i,\eta_i)=\left(\frac{i}{n},\frac{j}{n}\right)$ deb qaraymiz. U holda

$$\sigma=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(\xi_i,\eta_j)D_{ij}=\frac{1}{n^4}\sum_{i=0}^{n-1} i\sum_{j=0}^{n-1} j=\frac{1}{n^4}\cdot\frac{n(n-1)}{2}\cdot\frac{n(n-1)}{2}\Rightarrow \lim_{n\to\infty}\sigma=\frac{1}{4}.$$

bo‘ladi.

19.2. Darbu yig‘indilari

$f(x,y)$ funksiya $(D)\subset R^2$ sohada berilgan va chegaralangan bo‘lsin. (D) sohaning biror P bo‘linishini qaraymiz.

$$m_k=\inf_{(x,y)\in D_k}\{f(x,y)\},\quad M_k=\sup_{(x,y)\in D_k}\{f(x,y)\}$$

lar yordamida

$$s=\sum_{k=1}^{n} m_k D_k,\quad S=\sum_{k=1}^{n} M_k D_k$$

yig‘indilarni tuzamiz. Odatda bu yig‘indilar mos ravishda ***Darbuning quyi*** va ***yuqori yig‘indilari*** deb ataladi. (D) sohaning har bir bo‘linishiga nisbatan $\{s\},\{S\}$ to‘plamlarning chegaralanganligini va $s\le\sigma\le S$ munosabat o‘rinliligini ko‘rish qiyin emas.

Ta’rif.

$$\sup\{s\}=\underline{I},\quad \inf\{S\}=\bar{I}$$

miqdorlar mos ravishda $f(x,y)$ funksiyaning (D) sohadagi ***quyi ikki karrali*** hamda ***yuqori ikki karrali*** integrali deb ataladi.

Ta’rif. Agar $f(x,y)$ funksiyaning (D) sohada quyi hamda yuqori ikki karrali integrallari bir-biriga teng bo‘lsa, u holda $f(x,y)$ funksiya (D) sohada integrallanuvchi, ularning umumiy qiymati

$$I=\underline{I}=\bar{I}$$

$f(x,y)$ funksiyaning (D) sohadagi ***ikki karrali integrali*** (***Riman integrali***) deyiladi va

$$\iint\limits_{(D)} f(x,y)dD \qquad \left(\iint\limits_{(D)} f(x,y)dxdy\right)$$

kabi belgilanadi (19.1 dagi ta'rifga qarang).

19.3. Ikki karrali integralning mavjudligi

Teorema. $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lishi uchun, ixtiyoriy $\varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topilib, (D) sohaning diametri $\lambda < \delta$ bo'lgan har qanday bo'linishga nisbatan Darbu yig'indilari

$$S(f) - s(f) < \varepsilon$$

tengsizlikni qanoatlantirishi zarur va yetarli.

Teorema. Agar $f(x,y)$ funksiya (D) sohada chegaralangan va bu sohaning chekli sondagi nol yuzali chiziqlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, funksiya (D) sohada integrallanuvchi bo'ladi.

19.4. Ikki karrali integralning xossalari

1^0. $f(x,y)$ funksiya (D) sohada integrallanuvchi va (D) sohaga tegishli bo'lgan nol yuzali L chiziqdagi $(L \in (D))$ qiymatlarinigina o'zgartirishdan hosil bo'lgan $F(x,y)$ funksiya ham (D) sohada integrallanuvchi bo'lib,

$$\iint\limits_{(D)} f(x,y)dD = \iint\limits_{(D)} F(x,y)dD$$

bo'ladi.

2^0. $f(x,y)$ funksiya (D) sohada berilgan bo'lib, (D) nol yuzali L chiziq bilan (D_1) va (D_2) sohalarga ajratilgan bo'lsin.

$$\iint\limits_{(D)} f(x,y)dD = \iint\limits_{(D_1)} f(x,y)dD_1 + \iint\limits_{(D_2)} f(x,y)dD_2$$

munosabat o'rinli bo'ladi.

3^0. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $cf(x,y)$ $(c = const)$ ham shu sohada integrallanuvchi va

$$\iint\limits_{(D)} c\cdot f(x,y)dD = c\iint\limits_{(D)} f(x,y)dD$$

formula o'rinli bo'ladi.

4^0. Agar $f(x,y)$ va $g(x,y)$ funksiyalar (D) sohada integrallanuvchi bo'lsa, u holda $f(x,y) \pm g(x,y)$ funksiya ham shu sohada integrallanuvchi va

$$\iint_{(D)}[f(x,y)\pm g(x,y)]dD=\iint_{(D)}f(x,y)dD\pm\iint_{(D)}g(x,y)dD$$

formula o‘rinli bo‘ladi.

5^0. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo‘lib, ixtiyoriy $(x,y)\in(D)$ uchun $f(x,y)\geq 0$ bo‘lsa, u holda

$$\iint_{(D)}f(x,y)dD\geq 0$$

bo‘ladi.

6^0. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo‘lsa, u holda $|f(x,y)|$ funksiya ham shu sohada integrallanuvchi va

$$\left|\iint_{(D)}f(x,y)dD\right|\leq\iint_{(D)}|f(x,y)|dD$$

tengsizlik o‘rinli bo‘ladi.

7^0. *O‘rta qiymat haqida teorema.* Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo‘lsa, u holda shunday o‘zgarmas son

$$\mu\left(m\leq\mu\leq M, M=\sup_{(x,y)\in(D)}\{f(x,y)\}, m=\inf_{(x,y)\in(D)}\{f(x,y)\}\right)$$

mavjudki,

$$\iint_{(D)}f(x,y)dD=\mu D$$

formula o‘rinli bo‘ladi, bu yerda D (D) sohaning yuzi.

8^0. *O‘rta qiymat haqidagi umumlashgan teorema.* Agar $g(x,y)$ funksiya (D) sohada integrallanuvchi bo‘lib, shu sohada o‘z ishorasini saqlasa va $f(x,y)$ funksiya (D) sohada uzluksiz bo‘lsa, u holda shunday $(a,b)\in(D)$ topiladiki,

$$\iint_{(D)}f(x,y)g(x,y)dD=f(a,b)\iint_{(D)}g(x,y)dD$$

bo‘ladi.

19.5. Ikki karrali integrallarni hisoblash

(D) soha ushbu

$$(D)=\{(x,y)\in R^2: a\leq x\leq b, \varphi_1(x)\leq y\leq\varphi_2(x)\},$$
$$(\varphi_i(x)\in C[a,b], i=1,2)$$

ko‘rinishda bo‘lsin.

Teorema. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo‘lsin. Agar $x\in[a,b]$ da o‘zgaruvchining har bir tayin qiymatida

$$I(x)=\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

integral mavjud bo‘lsa, u holda ushbu

$$\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx$$

integral ham mavjud bo‘lib,

$$\iint_{(D)} f(x,y)dD=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx$$

o‘rinli bo‘ladi.

Agar $f(x,y)$ funksiya $(D)=\{(x,y)\in R^2 : a\le x\le b, c\le y\le d\}$ sohada berilgan va uzluksiz bo‘lsa,

$$\iint_{(D)} f(x,y)dD=\int_a^b\left[\int_c^d f(x,y)dy\right]dx=\int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

o‘rinli bo‘ladi.

19.6. Ikki karrali integrallarda o‘zgaruvchilarni almashtirish

$f(x,y)$ funksiya (D) sohada berilgan va uning chekli karrali integrali

$$\iint_{(D)} f(x,y)dxdy$$

mavjud bo‘lsin. Bu integralda o‘zgaruvchlarni quyidagicha almashtiramiz:

$$\begin{cases} x=\varphi(u,v) \\ y=\psi(u,v), \end{cases} (u,v)\in\Delta\in R^2. \qquad (2)$$

(2) quyidagi shartlarni qanoatlantirsin:

1°. (Δ) ni (D) ga o‘zaro bir qiymatli akslantiradi.

2°. $\varphi(u,v),\psi(u,v)$ funksiyalar (Δ) sohada uzluksiz, barcha xususiy hosilalarga ega va bu xususiy hosilalar ham uzluksiz.

$f(x,y)$ funksiya (D) sohada berilgan va uzluksiz bo‘lib, (2) akslantirish 1°-2° larni qanoatlantirsin. U holda

$$\iint_{(D)} f(x,y)dxdy=\iint_{(\Delta)} f(\varphi(u,v),\psi(u,v))|I(u,v)|dudv \qquad (3)$$

formula o‘rinli, bu yerda $I(u,v)=\dfrac{D(x,y)}{D(u,v)}=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}\neq 0$ (2) ning ***yakobianidir.***

(3) formula ikki karrali integrallarda o'zgaruvchlarni almashtirish formulasi deyiladi.

Misol. Ushbu

$$I=\iint\limits_{(D)}\left[1-\left(\frac{x}{a}\right)^{3/2}-\left(\frac{y}{b}\right)^{3}\right]dxdy$$

integralni hisoblang. Bu yerda

$$(D)=\left\{(x,y)\geq R^2 : x\geq 0, y\geq 0, \left(\frac{x}{a}\right)^{3/2}+\left(\frac{y}{b}\right)^{3}\leq 1\right\}.$$

Quyidagi

$$\frac{x}{a}=u^{2/3},\ \frac{y}{b}=v^{1/3}$$

almashtirishni bajaramiz. Qaralayotgan sohaning obrazi quyidagicha bo'ladi:

$$(\Delta)=\{(u,v)\in R^2 : u\geq 0, v\geq 0, u+v\leq 1\}$$

bo'ladi. Yakobiani esa:

$$I(u,v)=\frac{2ab}{9}u^{1/3}v^{-2/3}$$

ga teng.

$$I=\iint\limits_{(D)}\left(1-\left(\frac{x}{a}\right)^{3/2}-\left(\frac{y}{b}\right)^{3}\right)dxdy=\iint\limits_{(\Delta)}\frac{2ab}{9}(1-u-v)\cdot u^{-1/3}v^{-2/3}dudv=$$

$$=\frac{2ab}{9}\int_0^1 u^{-1/3}du\int_0^{1-u}(1-u-v)\cdot v^{-2/3}dv=\frac{2ab}{9}\int_0^1\frac{9}{4}(1-u)^{4/3}u^{-1/3}du=$$

$$=\frac{ab}{2}\cdot\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{2\sqrt{3}}{27}\pi ab$$

19.7. Uch karrali integrallar

$f(x,y,z)$ funksiya R^3 fazodagi chegaralangan (V) sohada berilgan bo'lsin. (V) sohaning P bo'linishini qaraylik. Bu bo'linishning har bir (V_k) ($k=1,2,3...,n$) bo'lagidan ixtiyoriy (ξ_k,η_k,ζ_k) nuqta olib, quyidagi

$$\sigma=\sum_{k=1}^{n} f\ (\xi_k,\eta_k,\zeta_k)V_k$$

integral yig'indini tuzamiz.

Ta'rif. Ixtiyoriy $\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilsaki, (V) sohaning diametri $\lambda<\delta$ bo'lgan har qanday bo'linishda hamda har bir (V_k) bo'lakdagi ixtiyoriy (ξ_k,η_k,ζ_k) nuqtalar uchun

$$|\sigma-I|<\varepsilon$$

tengsizlik bajarilsa, u holda I ga $f(x,y,z)$ funksiyaning (V) bo‘yicha uch karrali integrali deyiladi va

$$\iiint_{(V)} f(x,y,z)dV \qquad \left(\iiint_{(V)} f(x,y,z)dxdydz\right)$$

kabi belgilanadi.

Endi (V) soha-pastdan $z_1=\varphi_i(x,y)$, yuqoridan $z_2=\varphi_i(x,y)$ sirtlar bilan, yon tomondan Oz o‘qiga parallel silindrik sirt bilan chegaralangan soha bo‘lsin. Bu sohaning Oxy tekisligiga proyeksiyasi (D) bo‘lsin.

Agar $f(x,y,z)$ funksiya (V) sohada uzluksiz bo‘lib, $z_i=\varphi_i(x,y)$ ($i=1,2$) funksiyalar (D) da uzluksiz bo‘lsa, u holda

$$\iiint_{(V)} f(x,y,z)dxdydz=\iint_{(D)}\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y,z)dz\right)dxdy$$

bo‘ladi.

Agar

$$(D)=\{(x,y)\in R^2: a\le x\le b, \varphi_1(x)\le y\le \varphi_2(x)\}$$

bo‘lib, $\varphi_i(x)$ ($i=1,2$) funksiyalar $[a,b]$ da uzluksiz bo‘lsa, u holda

$$\iiint_{(V)} f(x,y,z)dxdydz=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)dz\right)dy\right]dx$$

bo‘ladi.

Misol. Ushbu

$$x^2+y^2+z^2=2az, \quad x^2+y^2=z^2, \quad x^2+y^2=\frac{1}{3}z^2$$

sirtlar bilan chegaralangan sohaning hajmini toping.

Izlanayotgan hajm

$$V=\iiint_{(V)} dxdydz$$

formula orqali topilib, bunda (V) yuqorida berilgan sirtlar bilan chegaralangandir.

Sferik koordinatalar sistemasidan foydalanamiz:

$$V=\iiint_{(V)} dxdydz \iiint_{(\Delta)} \rho^2\sin\theta d\rho d\varphi d\theta$$

$$(\Delta)=\left\{(\rho,\varphi,\theta): 0\le\varphi<2\pi,\ \frac{\pi}{6}\le\theta\le\frac{\pi}{4},\ 0\le r\le 2a\cos\theta\right\}.$$

Demak, bu yerda

$$V=\int_0^{2\pi}d\varphi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin\theta d\theta\int_0^{2a\cos\theta}\rho^2 d\rho=16\frac{\pi a^3}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos^3\theta\sin\theta d\theta=$$

$$=16\frac{\pi a^3}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos^3\theta d(\cos\theta)=\frac{5}{12}\pi\cdot a^3(kub\quad bir)$$

M14. Karrali integrallarga doir mashqlar

1. Takroriy integralni hisoblang:

1) $\int_0^1 dx\int_x^1(x+y)\,dy$ 2) $\int_1^2 dx\int_0^1 xy\,dy$

3) $\int_2^3 dx\int_1^2 x^2y\,dy$ 4) $\int_0^1 dx\int_0^2 x^2y^2\,dy$

5) $\int_1^2 dx\int_1^2 xy^2\,dy$ 6) $\int_1^2 dy\int_1^2 \frac{dx}{x^3}$

7) $\int_0^2 dy\int_0^1(x^2+2y)dx$ 8) $\int_{-3}^{3} dy\int_{y^2-4}^{5}(x+2y)dx$

9) $\int_3^4 dx\int_1^2\frac{dy}{(x+y)^2}$ 10) $\int_0^{2\pi}d\varphi\int_{a\sin\varphi}^{a}rdr$

11) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{3\cos\varphi}r^2\sin^2\varphi dr$ 12) $\int_0^1 dx\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}dy$

2. Quyidagi integrallarni integrallash tartibini o'zgartiring.

1) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{3\cos\varphi}r^2\sin^2\varphi dr$ 2) $\int_{-6}^{2}dx\int_{\frac{x^2}{4}-1}^{2-x}f(x,y)dy$

3) $\int_0^1 dx\int_{x^3}^{x}f(x,y)dy$ 4) $\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{1-x^2}f(x,y)dy$

5) $\int_{-1}^{1}dx\int_{x^{\frac{2}{3}}}^{-x^2+2}f(x,y)dy$ 6) $\int_0^1 dy\int_y^{2-y}f(x,y)dx$

3. Quyidagi integrallarni hisoblang.

1) $\iint\limits_{(D)}(x+y)dxdy$, $D=\{0\le x\le 1,\,1\le y\le 2\}$

2) $\iint\limits_{(D)}xy^3dxdy$, $D=\{0\le x\le 2,\,1\le y\le 3\}$

3) $\iint\limits_{(D)}(x^3y+xy^3)dxdy$, $D=\{(x,y)\in R^2: x\ge 0, y\ge 0, 4x^2-3y^2\le 4,\ 4y^2-3x^2\le 4\}$

4) $\iint\limits_{(D)}\frac{x^3-3xy^2+2y^3}{xy}dxdy$, D soha $y=\frac{1}{x^2}$, $y=\frac{4}{x^2}$, $y=x-1$, $y=x+1$ chiziqlar bilan chegaralangan.

5) $\iint\limits_{x^4+y^4\le 1}(x^2+y^2)dxdy$

4. Quyidagi chiziqlar bilan chegaralangan sohalar yuzalarini hisoblang (ikki karrali integral yordamida).

1) $y=2x$, $2x-y=7$, $x-4y=-7$, $x-4y=-14$

2) $y^2=x$, $y=x+2$, $y=2$, $y=-2$

3) $x^2+y^2=1$, $x+y=1$, $y=\frac{1}{2}$ $\left(x\ge 0,\ y\ge \frac{1}{2}\right)$

4) $y=(x-1)^2$, $y^2+x^2=1$

5) $y=4-x^2$, $3x-2y-6=0$

5. Qutb koordinatalarini kiritib quyidagi chiziqlar bilan chegaralangan shakl yuzini hisoblang.

1) $x^2+y^2-2ax=0$ va $x^2+y^2-2ay=0$

2) $x^2+y^2-2ax=0$ va $x^2+y^2-a^2y=0$

3) $x^2+y^2=r^2$ va $x^2+y^2-2ry=0,\ x=0$

4) $\left(x^2+y^2\right)^2=2a^2(x^2-y^2)$

6. Quyidagi sirtlar bilan chegaralangan jismlar hajmini hisoblang.

1) $y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

2) $z=y^2-x^2$, $y=-2$, $y=2$

3) $x^2+y^2=a^2$, $x+y+z=2a$, $z=0$

4) $x+y+z=4$, $x=3$, $x=0$, $y=0$, $y=2$, $z=0$

5) $y=x^2$, $y+z=2$, $z=0$

6) $z=3-x^2-y^2$, $z=0$

7) $y=x^2$, $z=x^2+y^2$, $z=0$, $y=1$

8) $z=x^2+y^2$, $x+y=1$, $x=0$, $y=0$, $z=0$

7. Qutb koordinatalar sistemasiga o'tish yo'li bilan quyidagi sirtlar bilan chegaralangan jismning hajmini hisoblang.

1) $z=4-x^2-y^2$, $z=\frac{2+x^2+y^2}{2}$

2) $x^2+y^2+z-4=0$, $z=0$

3) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a>0$, $b>0$, $c>0$

4) $z=x^2+y^2$, $x^2+y^2=x$, $x^2+y^2=2x$, $z=0$

5) $y=\sqrt{x}$, $y=2\sqrt{x}$, $x+z=0$, $z=0$

6) $z=x^2+y^2$, $y=x^2$, $y=1$, $z=0$.

8. Takroriy integralni hisoblang.

1) $\int_0^1 dx\int_0^3 dy\int_0^2 (x+y+z)dz$ 2) $\int_0^1 dx\int_0^1 dy\int_0^1 (3x+y+2z)dz$

3) $\int_0^2 dx \int_0^1 dy \int_0^{x+2y} dz$ 4) $\int_0^1 dx \int_0^x dy \int_0^{\sqrt{x^2+y^2}} zdz$

9. Uch karrali integralni hisoblang.

1) $\iiint_{(V)} (2x+y-z)dxdydz$, (V) $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

2) $\iiint_{(V)} (x+y+z)dxdydz$, (V) $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

3) $\iiint_{(V)} (3x+2y+z)dxdydz$, (V) $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$

4) $\iiint_{(V)} \frac{dxdydz}{(1+x+y+z)^4}$, (V) $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$

5) $\iiint_{(V)} xy^2z^3dxdydz$, (V) $z = xy$, $y = x$, $x = 1$, $z = 0$

6) $\iiint_{(V)} (x^2+y^2)dxdydz$, (V) $x^2+y^2 = 2z$, $z = 2$ sirtlar bilan chegaralangan.

10. Sferik koordinatalar kiritib, uch karrali integralni hisoblang.

1) $z = 0$

1) $\iiint_{(V)} x^2dxdydz$, $x^2+y^2+z^2 \le R^2$ shardan iborat.

2) $\iiint_{(V)} (x^2+y^2+z^2)dxdydz$, $x^2+y^2+z^2 \le R^2$ shardan iborat.

3) $\iiint_{(V)} (x^2+y^2)dxdydz$, $x^2+y^2+z^2 \le R^2$ $(z \ge 0)$ yarim shardan iborat.

4) $\iiint_{(V)} \sqrt{x^2+y^2+z^2}dxdydz$, $x^2+y^2+z^2 \le R^2$ shardan iborat.

Testlar

1) Qaysi belgi n-karrali integral belgisi ($G \subset R^n$).

1. $\int_G f(x)dx$; 2. $\overbrace{\iint_G \ldots \int}^{n-marta} f(x)dx_1dx_2..dx_n$; 3. $\int_G f(x_1, x_2, \ldots, x_n)dx_1dx_2..dx_n$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

2) $f(x)$ va $g(x)$ funksiyalar $G \subset R^n$ da integrallanuvchi, α, β o'zgarmas sonlar bo'lsa, to'g'ri tenglikni toping ($G_1 \cup G_2 = G$).

1. $\int_G (f(x)-g(x))dx = \int_G f(x)dx - \int_G g(x)dx$

2. $\int_G \alpha g(x)dx = \alpha \int_G g(x)dx$

3. $\left|\int_G f(x)dx\right| \le \int_G |f(x)|dx$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

3. $f(x)$ va $g(x)$ funksiyalar $G \subset R^n$ da integrallanuvchi, α, β o‘zgarmas sonlar bo’lsa, to‘g‘ri tenglikni toping ($G_1 \cup G_2 = G$).

1. $\int\limits_G f(x)dx = \int\limits_{G_1} f(x)dx + \int\limits_{G_2} f(x)dx$

2. $\int\limits_G f(x)dx = \int\limits_{G_1} f(x)dx - \int\limits_{G_2} f(x)dx$

3. $\int\limits_{G_1} f(x)dx = \int\limits_G f(x)dx - \int\limits_{G_2} f(x)dx$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

4. $f(x,y)$ funksiya $a \le x \le b$, $c \le y \le d$ sohada integrallanuvchi. $\int\limits_a^b f(x,y)dx$ $y \in [c,d]$ uchun $\int\limits_c^d f(x,y)dy$ $x \in [a,b]$ uchun integrallanuvchi bo’lsa, $\int\limits_a^b\int\limits_c^d f(x,y)dxdy$ ga teng ifodani toping.

1. $\int\limits_a^b dx \int\limits_d^c f(x,y)dy$ 2. $\int\limits_d^c dy \int\limits_a^b f(x,y)dx$ 3. $\int\limits_b^a dx \int\limits_d^c f(x,y)dy$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

5. $G = \{(x,y): 0 \le x \le 1,\ x^2 \le y \le 1\}$ bo‘lsa, $\iint\limits_G x^2 dxdy$ ni toping.

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

6. $y = x^2$, $y = 2x^2$, $xy = 1$, $xy = 2$ chiziqlar bilan chegaralangan sohada ikki karrali integralni hisoblash uchun qanday almashtirish bajarish kerak?

1. $\xi = yx^{-2}$, $\eta = xy$ 2. $\xi = y^{-2}x$, $\eta = xy$ 3. $\xi = yx^{-1}$, $\eta = xy$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

7. Qutb koordinatalar sistemasiga o’tishning yakobianini aniqlang.

1. r 2. $r^2 \sin\psi$ 3. $r^2 \cos\psi$

A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

8. $\iint\limits_{0<x^2+y^2<R^2} \frac{dxdy}{(x^2+y^2)^\alpha}$ integral yaqinlashuvchi bo’ladigan α ning qiymatini toping.

1. $\alpha < 1$ 2. $\alpha = 1$ 3. $\alpha > 1$

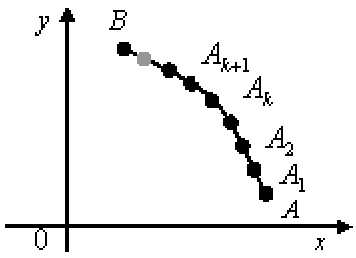
A) 1 B) 2 C) 3 D) 1,2 E) 1,2,3

20-§. EGRI CHIZIQLI INTEGRALLAR

20.1. Birinchi tur egri chiziqli integral

AB egri chiziqda $f(x,y)$ funksiya aniqlangan bo‘lsin.

Bu egri chiziqning $P=\{A_0, A_1, ..., A_n\}$ bo‘linishini va uning har bir $\overset{\smile}{A_k A_{k+1}}$ yoyida ixtiyoriy (ξ_k, η_k) nuqta ($k=0,1,2,\ldots,n-1$) olamiz va quyidagi



28-rasm

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta s_k \qquad (1)$$

yig‘indini tuzamiz. Δs_k $\overset{\smile}{A_k A_{k+1}}$ yoyning uzunligi. Odatda (1) ***integral yig‘indi*** deyiladi.

Ta’rif. Agar $\lambda_p \to 0$ da σ yig‘indi chekli limitga ega bo‘lsa, u holda $f(x,y)$ funksiya AB egri chiziq bo‘yicha integrallanuvchi, limit esa $f(x,y)$ funksiyaning ***birinchi tur egri chiziqli integrali*** deyiladi va

$$\int_{AB} f(x,y)ds$$

kabi belgilanadi:

$$\int_{AB} f(x,y)ds=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta s_k$$

Integralning mavjudligi. Faraz qilaylik AB egri chiziq ushbu

$$x=x(s),\ \ y=y(s) \qquad (0\le s\le S) \qquad (2)$$

sistema bilan berilgan bo‘lsin. Bunda $s-AQ$ yoyning uzunligi $Q=(x,y)\in AB$, S esa ABning uzunligi.

Teorema. Agar $f(x,y)$ funksiya AB egri chiziqda berilgan va uzluksiz bo‘lsa, u holda uning AB bo‘yicha birinchi tur egri chiziqli integrali mavjud va

$$\int_{AB} f(x,y)ds=\int_0^s f(x(s),y(s))ds$$

bo‘ladi.

Endi AB egri chiziq ushbu

$$x=\varphi(t),\ \ y=\psi(t)\ \ (\alpha\le t\le\beta) \qquad (3)$$

sistema bilan (parametrik formada) berilgan bo‘lsin.

Teorema. Agar $f(x,y)$ funksiya AB da berilgan va uzluksiz bo‘lsa, u holda uning AB bo‘yicha birinchi tur egri chiziqli integrali mavjud va

$$\int_{AB} f(x,y)ds = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t)) \cdot \sqrt{\varphi'^2(t) + \psi'^2(t)}dt \qquad (4)$$

bo‘ladi.

20.2. Birinchi tur egri chiziqli integralning xossalari

Agar $f(x,y)$ funksiyaning AB bo‘yicha birinchi tur egri chiziqli integrali mavjud bo‘lsa, u holda quyidagi xossalar o’rinli bo‘ladi:

1°. Agar $AB = AC \cup CB$ bo‘lsa, u holda

$$\int_{AB} f(x,y)ds = \int_{AC} f(x,y)ds + \int_{CB} f(x,y)ds$$

bo‘ladi.

2°. Ushbu

$$\int_{AB} C \cdot f(x,y)ds = C \int_{AB} f(x,y)ds \qquad \text{(C-const)}$$

tenglik o‘rinli.

3°. Quyidagi

$$\int_{AB} [f(x,y) \pm g(x,y)]ds = \int_{AB} f(x,y)ds \pm \int_{AB} g(x,y)ds$$

tenglik o‘rinli bo‘ladi.

4°. Agar ixtiyoriy $(x,y) \in AB$ da $f(x,y) \geq 0$ bo‘lsa, u holda

$$\int_{AB} f(x,y)ds \geq 0$$

bo‘ladi.

5°. $|f(x,y)|$ funksiya AB da integrallanuvchi bo‘lsa, u holda

$$\left| \int_{AB} f(x,y)ds \right| \leq \int_{AB} |f(x,y)|ds$$

bo‘ladi.

6°. Shunday $(c_1, c_2) \in AB$ nuqta topiladiki,

$$\int_{AB} f(x,y)ds = f(c_1, c_2) \cdot S$$

bo‘ladi, S bunda AB ning uzunligi.

Integralni hisoblash. Egri chiziqli integrallar Riman integrallariga keltirib hisoblanadi. Bunda

$$\int_{AB} f(x,y)ds = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\sqrt{\varphi^{`2}(t) + \psi^{`2}(t)}dt \qquad (5)$$

hamda quyida keltirilgan formulalardan foydalaniladi.

Aytaylik, AB egri chiziq ushbu

$$y = y(x) \qquad \left(a \leq x \leq b, \quad y(a) = A, \quad y(b) = B\right)$$

tenglama bilan aniqlangan bo‘lsin, u holda

$$\int_{AB} f(x,y)ds = \int_a^b f(x, y(x))\sqrt{1+y'^2(x)}dx \qquad (6)$$

bo‘ladi.

Endi AB egri chiziq ushbu

$$\rho = \rho(\theta) \ \ (\theta_0 \le \theta \le \theta_1)$$

tenglama bilan (qutb koordinata sistemasida) berilgan bo‘lib, $\rho(\theta)$ funksiya $[\theta_0, \theta_1]$ da uzluksiz $\rho'(\theta)$ hosilaga ega bo‘lsin. Agar $f(x,y)$ funksiya shu AB da berilgan va uzluksiz bo‘lsa, u holda

$$\int_{AB} f(x,y)ds = \int_{\theta_1}^{\theta_2} f(\rho\cos\theta, \rho\sin\theta)\sqrt{\rho^2+\rho'^2}d\theta \qquad (7)$$

bo‘ladi.

Misol. Ushbu

$$\int_{AB} (4\sqrt[3]{x} - 3\sqrt{y})ds$$

birinchi tur egri chiziqli integralni hisoblang, bunda AB tekislikning $A=(-1;0)$, B=(0,1) nuqtalarini birlashtiruvchi to‘g‘ri chiziq kesmasi.

Ravshanki, A va B nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi

$$y = x + 1$$

bo‘lib, berilgan integral esa

$$y = x+1, \quad -1 \le x \le 0$$

kesma bo‘yicha olingan integral bo‘ladi.

Unda (6) formulaga ko‘ra

$$\int_{AB} (4\sqrt[3]{x} - 3\sqrt{y})ds = \int_{-1}^{0} (4\sqrt[3]{x} - 3\sqrt{x+1})\sqrt{1+(x+1)'^2}dx$$

ga teng. Keyingi integralni hisoblaymiz:

$$\int_{-1}^{0} (4\sqrt[3]{x} - 3\sqrt{x+1})\sqrt{2}dx = \sqrt{2}\left[3\cdot x^{\frac{4}{3}} - 2(x+1)^{\frac{3}{2}}\right]_{-1}^{0} = -5\sqrt{2}$$

Demak,

$$\int_{AB} (4\sqrt[3]{x} - 3\sqrt{y})ds = -5\sqrt{2}$$

20.3. Birinchi tur egri chiziqli integral tatbiqlari.

Birinchi tur egri chiziqli integrallar yordamida yoy uzunligini, jismning massasini, og‘irlik markazlarini toppish mumkin.

1°. Tekislikda to‘g‘rilanuvchi AB egri chiziq berilgan bo‘lsin. Uning uzunligi ushbu

$$S=\int_{AB} ds$$

formula bilan topiladi.

2°. Tekislikda to‘g‘rilanuvchi AB egri chizig‘i bo‘yicha massa tarqatilgan bo‘lib, uning zichligi $\rho=\rho(x,y)$ bo‘lsin. Bu egri chiziqning massasi

$$m=\int_{AB}\rho(x,y)ds\,,$$

og‘irlik markazining koordinatalari esa

$$x_0=\frac{1}{m}\int_{AB}x\cdot\rho(x,y)ds,\ y_0=\frac{1}{m}\int_{AB}y\cdot\rho(x,y)ds$$

formula orqali topiladi.

20.4. Ikkinchi tur egri chiziqli integrallar xossalari

AB egri chiziqning $P=\{A_0,A_1,...,A_n\}$ bo‘linishini va uning har bir $A_k\breve{A}_{k+1}$ yoyida ixtiyoriy (ξ_k,η_k) nuqta olib funksiyaning shu nuqtadagi qiymati $f(\xi_k,\eta_k)$ ni $A_k\breve{A}_{k+1}$ ning OX (OY) o‘qidagi $\Delta x_k(\Delta y_k)$ proyeksiyasiga ko‘paytirib, quyidagi yig‘indini tuzamiz:

$$\sigma'=\sum_{k=0}^{n-1}f(\xi_k,\eta_k)\Delta x_k \qquad \left(\sigma''=\sum_{k=0}^{n-1}f(\xi_k,\eta_k)\Delta y_k\right)$$

Ta’rif. Agar $\lambda_p\to 0$ da σ' yig‘indi (σ'' yig‘indi) chekli limitga ega bo‘lsa, u holda $f(x,y)$ funksiya AB egri chiziq bo‘yicha integrallanuvchi deyiladi. Bu limit $f(x,y)$ funksiyaning ikkinchi tur egri chiziqli integrali deyiladi va

$$\int_{AB}f(x,y)dx \quad \left(\int_{AB}f(x,y)dy\right)$$

kabi belgilanadi.

Integralning mavjudligi. Faraz qilaylik, AB ushbu

$$x=\varphi(t),\quad y=\psi(t)\quad (\alpha\le t\le\beta)$$

sistema orqali berilgan bo‘lsin.

Teorema. Agar $f(x,y)$ funksiya AB egri chiziqda berilgan va uzluksiz bo‘lsa, u holda uning AB bo‘yicha ikkinchi tur egri chiziqli integrali mavjud bo‘lib

$$\int_{AB}f(x,y)dx=\int_{\alpha}^{\beta}f(\varphi(t),\psi(t))\varphi'(t)dt \qquad \left(\int_{AB}f(x,y)dy=\int_{\alpha}^{\beta}f(\varphi(t),\psi(t))\psi'(t)dt\right)$$

bo‘ladi.

Teorema. Agar $P(x,y)$ va $Q(x,y)$ funksiyalar AB da berilgan va uzluksiz bo‘lsa, u holda bu funksiyalarning ikkinchi tur egri chiziqli integrallari mavjud va

$$\int_{AB} P(x,y)dx + Q(x,y)dy = \int [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]dt$$

bo‘ladi.

20.5. Ikkinchi tur egri chiziqli integrallar xossalari

1°. Ikkinchi tur egri chiziqli integrallar integrallash egri chizig‘ining yo‘nalishiga bog‘liq bo‘ladi:

$$\int_{BA} f(x,y)dx = -\int_{AB} f(x,y)dx; \qquad \int_{BA} f(x,y)dy = -\int_{AB} f(x,y)dy.$$

2°. Agar AB egri chiziq OX (OY) o‘qiga perpendikulyar bo‘lgan to‘g‘ri chiziq kesmasidan iborat bo‘lsa, u holda

$$\int_{AB} f(x,y)dx = 0 \quad \left(\int_{AB} f(x,y)dy = 0\right)$$

3°. Agar $f(x,y)$ funksiya AB da integrallanuvchi bo‘lib, $AB = AC \cup CB$ bo‘lsa,

$$\int_{AB} f(x,y)dx = \int_{AC} f(x,y)dx + \int_{CB} f(x,y)dx$$

bo‘ladi.

4°. Agar $f(x,y)$ funksiya AB da integrallanuvchi bo‘lsa, u holda

$$\int_{AB} kf(x,y)dx = k\int_{AB} f(x,y)dx$$

bo‘ladi, bunda $k = const$.

5°. Agar $f(x,y)$ va $g(x,y)$ funksiyalar AB da integrallanuvchi bo‘lsa, u holda

$$\int_{AB} [f(x,y) \pm g(x,y)]dx = \int_{AB} f(x,y)dx \pm \int_{AB} g(x,y)dx$$

bo‘ladi.

Misol. Ushbu

$$\int_{AB} y^2dx + x^2dy$$

integralni hisoblang, bunda AB egri chiziq $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning yuqori yarim tekislikdagi qismidan iborat.

Bu ellipsning parametrik tenglamasini yozamiz:

$$x = a\cos t$$
$$y = b\sin t$$

$A=(a,0)$ nuqtaga parametrning $t=0$ qiymati $B=(-a,0)$ nuqtaga esa $t=\pi$ qiymati mos kelib, t parametr 0 dan π gacha o‘zgarganda (x,y) nuqta A dan B ga qarab ellipsning yuqori yarim tekislikdagi qismini chizadi.

$$P(x,y)=y^2,\quad Q(x,y)=x^2$$

funksiyalar esa AB da uzluksiz.

$$\int_{AB} y^2dx+x^2dy=\int_0^{\pi}\left[b^2\sin^2 t\cdot(-a\sin t)+a^2\cos^2 t\cdot b\cos t\right]dt=$$

$$ab\int_0^{\pi}(a\cos^3 t-b\sin^3 t)dt=-\frac{4}{3}ab^2$$

20.6. Grin formulasi

Teorema. $P(x,y)$ funksiya $\overline{D}$ $(D=\overline{D}\cup\partial D)$ sohada berilgan va uzluksiz bo‘lsin. Agar bu funksiya D sohada uzluksiz $\frac{\partial P(x,y)}{\partial y}$ xususiy hosilaga ega bo‘lsa, u holda

$$\int_{\partial D} P(x,y)dx=-\iint_D \frac{\partial P(x,y)}{\partial y}dxdy$$

bo‘ladi.

$P(x,y)$ va $Q(x,y)$ funksiyalar D da berilgan va uzluksiz bo‘lsin. Agar bu funksiyalar D da uzluksiz $\frac{\partial P(x,y)}{\partial y},\frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega bo‘lsa, u holda quyidagi Grin formulasi o‘rinli:

$$\int_{\partial F} P(x,y)dx+Q(x,y)dy=\iint_F\left(\frac{\partial Q(x,y)}{\partial x}-\frac{\partial P(x,y)}{\partial y}\right)dxdy.$$

Teorema. $Q(x,y)$ funksiya $\overline{G}$ sohada berilgan va uzluksiz bo‘lsin. Agar bu funksiya G sohada uzluksiz $\frac{\partial Q(x,y)}{\partial x}$ xususiy hosilaga ega bo‘lsa, u holda

$$\int_{\partial G} Q(x,y)dy=\iint_G \frac{\partial Q(x,y)}{\partial x}dxdy$$

bo‘ladi.

Quyidagi tasdiqlar o‘rinli:

1°. Agar D sohada $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$ bo‘lsa, u holda D sohaga tegishli bo‘lgan har qanday K yopiq chiziq bo‘yicha olingan integral

$$\oint_K P(x,y)dx+Q(x,y)dy=0$$

bo‘ladi.

2°. Agar D da tegishli bo'lgan har qanday K yopiq chiziq bo'yicha olingan integral uchun

$$\oint_K P(x,y)dx + Q(x,y)dy = 0$$

bo'lsa, u holda

$$\int_{AB} P(x,y)dx + Q(x,y)dy \qquad (AB \subset D)$$

integral A va B nuqtalarni birlashtiruvchi egri chiziqqa bog'liq bo'lmaydi.

3°. Agar ushbu

$$\int_{AB} P(x,y)dx + Q(x,y)dy \qquad (AB \subset D)$$

integral A va B nuqtalarni birlashtiruvchi egri chiziqqa bog'liq bo'lmasa, u holda

$$P(x,y)dx + Q(x,y)dy$$

ifoda D sohada berilgan biror funksiyaning to'liq differensiali bo'ladi.

4°. Agar

$$P(x,y)dx + Q(x,y)dy$$

ifoda D sohada berilgan biror funksiyaning to'liq differensiali bo'lsa, u holda

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

bo'ladi.

Misol. Ushbu

$$\oint_k (3x^2 + y)dx + (x - 2y^2)dy = 0$$

tenglikning o'rinli bo'lishini isbotlang, bunda K egri chiziq uchlari $(0;0)$, $(1;0)$, $(0;1)$ nuqtalarda bo'lgan uchburchak konturdan iborat.

Berilgan integralda

$$P(x,y) = 3x^2 + y, \quad Q(x,y) = x - 2y^2.$$

Bu funksiyalar tekislikda uzluksiz hamda

$$\frac{\partial P(x,y)}{\partial y} = 1, \quad \frac{\partial Q(x,y)}{\partial x} = 1,$$

uzluksiz xususiy hosilalarga ega bo'lib,

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}.$$

Unda yuqoridagi 1^0-tasdiqqa binoan $P(x,y)dx + Q(x,y)dy$ ning yopiq kontur bo'yicha (berilgan uchburchak konturi bo'yicha) integrali nolga teng bo'ladi:

$$\oint_k \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy = 0$$

20.7. Sirt integrali

Fazoda ushbu

$$z = z(x, y) \qquad (1)$$

tenglama bilan aniqlangan (S) sirt berilgan bo‘lsin. Bunda $z(x, y)$ funksiya (D) sohada $\left((D) \subset R^2\right)$ berilgan funksiya bo‘lib, u shu sohada uzluksiz $z'_x(x, y), z'_y(x, y)$ hosilalarga ega.

Ma`lumki, bunday sirt yuzaga ega bo‘lib, quyidagi

$$S = \iint_{(D)} \sqrt{1 + z'^2_x(x, y) + z'^2_y(x, y)}dxdy$$

formula orqali hisoblanadi.

20.8. Birinchi tur sirt integrallari

$f(x, y, z)$ funksiya (S) sirtda $\left((S) \subset R^3\right)$ berilgan bo‘lsin. Bu sirtning P bo‘linishini va bu bo‘linishning har bir (S_k) bo‘lagida ($k = 1,2,\ldots,n$) ixtiyoriy (ξ_k, η_k, ζ_k) nuqtani olaylik. Berilgan funksiyaning (ξ_k, η_k, ζ_k) nuqtadagi qiymati $f(\xi_k, \eta_k, \zeta_k)$ ni (S_k) sirtning S_k yuziga ko‘paytirib, quyidagi yig‘indini tuzamiz :

$$\sigma = \sum_{k=1}^{n} f\ (\xi_k, \eta_k, \zeta_k)\, S_k \qquad (1)$$

Odatda (1) integral yig‘indi deyiladi.

Ta’rif. Agar $\lambda_p \to 0$ da $f(x, y, z)$ funksiyaning integral yig‘indisi σ chekli limitga ega bo‘lsa, $f(x, y, z)$ funksiya (S) sirt bo‘yicha integrallanuvchi deyiladi. Bu yig‘indining limiti I esa $f(x, y, z)$ funksiyaning ***birinchi tur sirt integrali*** deyiladi va

$$\iint_{(S)} f(x, y, z)ds$$

kabi belgilanadi:

$$\iint_{(S)} f(x, y, z)ds = \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f\ (\xi_k, \eta_k, \zeta_k)\, S_k$$

20.9. Birinchi tur sirt integralining mavjudligi

Teorema. Agar $f(x, y, z)$ funksiya (S) sirtda berilgan va uzluksiz bo‘lsa, u holda uning (S) sirt bo‘yicha birinchi tur sirt integrali

$$\iint_{(S)} f(x, y, z)ds$$

mavjud bo‘lib

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x,y,z(x,y)) \times \sqrt{1 + z_x'^2(x,y) + z_y'^2(x,y)}dxdy \qquad (1)$$

bo‘ladi.

Teorema. Agar $f(x,y,z)$ funksiya (S) sirtda berilgan va uzluksiz bo‘lsa, u holda uning (S) sirt bo‘yicha birinchi tur sirt integrali

$$\iint\limits_{(S)} f(x,y,z)ds$$

mavjud bo‘lib,

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x(y,z),y,z) \times \sqrt{1 + x_x'^2(y,z) + x_y'^2(y,z)}dydz$$

bo‘ladi.

20.10. Birinchi tur sirt integralining xossalari

1^0. Agar $f(x,y,z)$ funksiya (S) sirt bo‘yicha integrallanuvchi bo‘lib, $(S) = (S_1) \cup (S_2)$ bo‘lsa, u holda

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(S_1)} f(x,y,z)ds + \iint\limits_{(S_2)} f(x,y,z)ds$$

bo‘ladi.

2^0. Agar $f(x,y,z)$ funksiya (S) sirt bo‘yicha integrallanuvchi bo‘lsa, u holda $cf(x,y,z)$ ham ($c = const$) shu sirt bo‘yicha integrallanuvchi bo‘ladi va

$$\iint\limits_{(S)} c \cdot f(x,y,z)ds = c \cdot \iint\limits_{(S)} f(x,y,z)ds$$

tenglik o‘rinli bo‘ladi.

3^0. Agar $f(x,y,z)$ va $g(x,y,z)$ funksiyalarning har biri (S) sirt bo‘yicha integrallanuvchi bo‘lsa, u holda $f(x,y,z) \pm g(x,y,z)$ ham shu sirt bo‘yicha integrallanuvchi bo‘lib,

$$\iint\limits_{(S)} [f(x,y,z) \pm g(x,y,z)]ds = \iint\limits_{(S)} f(x,y,z)ds \pm \iint\limits_{(S)} g(x,y,z)ds$$

bo‘ladi.

Misol. Ushbu

$$\iint\limits_{(S)} |xyz|ds$$

integralni hisoblang, bunda (S) sirt $z = x^2 + y^2$ aylanma paraboloidning $z = 0, z = 1$ tekisliklar orasidagi qismi.

Ravshanki, (S) sirtning OXY tekislikdagi proyeksiyasi

$$(D) = \{(x,y) \in R^2 : x^2 + y^2 \le 1\}$$

doiradan iborat bo‘ladi.

(3) formuladan foydalanib topamiz:

$$\iint_{(S)} |xyz|ds = \iint_{(D)} \left|xy(x^2+y^2)\right|\sqrt{1+z_x'^2+z_y'^2}\,dxdy = \iint_{(D)} (x^2+y^2)|xy|\sqrt{1+4(x'^2+y'^2)}\,dxdy\,.$$

Ikki karrali integralni hisoblashda quyidagi almashtirishni bajaramiz:

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi \qquad (0\le\rho\le 1,0\le\varphi\le 2\pi)\,.$$

Natijada

$$\iint_{(D)} (x^2+y^2)|xy|\sqrt{1+4(x'^2+y'^2)}\,dxdy =$$

$$4\int_0^{\frac{\pi}{2}}\left(\int_0^1\left(\rho\cos\varphi\cdot\rho\sin\varphi\cdot\rho^2\sqrt{1+\rho^2}\right)\rho d\rho\right)d\varphi = 2\int_0^1 \rho^5\cdot\sqrt{1+4\rho^2}\,d\rho = \frac{125\sqrt{5}-1}{420}$$

bo‘ladi. Demak,

$$\iint_{(S)} |xyz|ds = \frac{125\sqrt{5}-1}{420}\,.$$

20. 11. Birinchi tur sirt integralining ba’zi bir tatbiqlari

1⁰. (S) sirtning yuzi

$$S=\iint_{(S)} ds$$

formula bilan topiladi.

2⁰. Agar (S) sirt bo‘yicha zichligi $\rho(x,y,z)$ bo‘lgan massa tarqatilgan bo‘lsa, unda (S) sirtning massasi

$$m=\iint_{(S)} \rho(x,y,z)ds$$

formula orqali topiladi.

3⁰. (S) sirtning og‘irlik markazining koordinatalari

$$x_0=\frac{1}{m}\iint_{(S)} x\rho(x,y,z)ds\ ,\quad y_0=\frac{1}{m}\iint_{(S)} y\rho(x,y,z)ds\,,\quad z_0=\frac{1}{m}\iint_{(S)} z\rho(x,y,z)ds$$

bo‘ladi.

4⁰. (S) sirtning OX, OY, OZ koordinata o‘qlariga nisbatan inertsiya momentlari mos ravishda ushbu

$$I_x=\iint_{(S)} (z^2+y^2)\rho(x,y,z)ds,\quad I_y=\iint_{(S)} (x^2+y^2)\rho(x,y,z)ds,\quad I_z=\iint_{(S)} (z^2+x^2)\rho(x,y,z)ds$$

formulalar bilan topiladi.

(S) sirtning OX, OY, OZ koordinata tekisliklariga nisbatan inertsiya momentlari mos ravishda quyidagicha bo‘ladi:

$$I_{xy}=\iint_{(S)} z^2\rho(x,y,z)ds,\quad I_{xz}=\iint_{(S)} y^2\rho(x,y,z)ds,\quad I_{yz}=\iint_{(S)} x^2\rho(x,y,z)ds.$$

20.12. Ikkinchi tur sirt integrallari.

$f(x,y,z)$ funksiya (S) sirtda berilgan bo‘lsin. Bu sirtning p bo‘linishini va bu bo‘linishning har bir (S_k) bo‘ligida $(k=1,2,...,n)$ ixtiyoriy (ξ_k,η_k,ζ_k) nuqtani olaylik. Berilgan funksiyaning (ξ_k,η_k,ζ_k) nuqtadagi $f(\xi_k,\eta_k,\zeta_k)$ qiymatini OXY (OYZ, OZX) tekislikdagi proyeksiyasi (D_k) $((D_k'),(D_k''))$ ning yuziga ko‘paytirib, quyidagi integral yig‘indini tuzamiz :

$$\sigma=\sum_{k=1}^{n} f\ (\xi_k,\eta_k,\zeta_k)\cdot D_k$$

$$\left(\sigma'=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\,D_k',\sigma''=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\,D_k''\right)$$

Ta’rif. Agar $\lambda_p\to 0$ da $f(x,y,z)$ funksiyaning integral yig‘indisi σ (σ',σ'') chekli limitga ega bo‘lsa, $f(x,y,z)$ funksiya (S) sirtning tanlagan tomoni bo‘yicha integrallanuvchi deyiladi. Bu yig‘indining limiti I esa (I',I''), $f(x,y,z)$ funksiyaning (S) sirtning tanlangan tomoni bo‘yicha ikkinchi tur integrali deyiladi va

$$\iint\limits_{(S)} f(x,y,z)dxdy \quad \left(\iint\limits_{(S)} f(x,y,z)dydz, \iint\limits_{(S)} f(x,y,z)dzdx\right)$$

kabi belgilanadi:

$$\iint\limits_{(S)} f(x,y,z)dxdy=\lim_{\lambda_p\to 0}\sum_{k=1}^{n} f\ (\xi_k,\eta_k,\zeta_k)\cdot D_k$$

$$\left(\iint\limits_{(S)} f(x,y,z)dydz=\lim_{\lambda_p\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k',\quad \iint\limits_{(S)} f(x,y,z)dzdx=\lim_{\lambda_p\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k''\right)$$

20.13. Ikkinchi tur sirt integralning mavjudligi

Teorema. Agar $f(x,y,z)$ funksiya (S) sirtda berilgan va uzluksiz bo‘lsa, u holda uning (S) sirt bo‘yicha olingan ikkinchi tur sirt integrali

$$\iint\limits_{(S)} f(x,y,z)dxdy$$

mavjud bo‘lib,

$$\iint\limits_{(S)} f(x,y,z)dydz=\iint\limits_{(D)} f(x(y,z),y,z)dydz$$

bo‘ladi.

20.14. Ikkinchi tur integralning xossalari

1^0. Funksiyaning (S) sirtning bir tomoni bo‘yicha olingan ikkinchi tur sirt integrali, funksiyaning shu sirtning ikkinchi tomoni bo‘yicha olingan ikkinchi tur sirt integralidan faqat ishorasi bilan farq qiladi.

2^0. $f(x,y,z)$ funksiyaning yasovchilari OZ o‘qiga parallel bo‘lgan silindrik (S) sirt bo‘yicha ikkinchi tur sirt integrali

$$\iint_{(S)} f(x,y,z)dxdy$$

uchun

$$\iint_{(S)} f(x,y,z)dxdy = 0$$

bo‘ladi.

Integralni hisoblash. Ikkinchi tur sirt integrallari ikki karrali integrallarga keltirib hisoblanadi:

$$\iint_{(s)} f(x,y,z)dxdy = \iint_{(S)} f(x,y,z(x,y))dxdy, \quad (1)$$

$$\iint_{(S)} f(x,y,z)dydz = \iint_{(S)} f(x(y,z),y,z)dydz, \quad (2)$$

$$\iint_{(S)} f(x,y,z)dzdx = \iint f(x,y(z,x),z)dzdx. \quad (3)$$

Misol. Ushbu

$$\iint_{(S)} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy$$

integralni hisoblang, bunda (S) sirt

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

ellipsoidning $z=0$ tekislikdan pastda joylashgan qismi bo‘lib, integral shu sirtning pastki tomoni bo‘yicha olingan.

Ma’lumki, (S) sirtning tenglamasi

$$z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

bo‘lib, uning OXY tekislikdagi proyeksiyasi

$$(D)=\left\{(x,y)\in R^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

bo‘ladi.

(1) formuladan foydalanib,

$$\iint_{(S)} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy = -\iint_{(D)} \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dxdy$$

bo‘lishini topamiz. Integral (S) sirtning pastki tomoni bo‘yicha olinganligi sababli sirt integrali minus ishora bilan olinadi.

Endi

$$\iint_{(D)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dxdy=\iint_{(D)}\left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy$$

ikki karrali integralni hisoblaymiz. Bu integralda o‘zgaruvchilarni $x=a\rho\cos\varphi, y=b\rho\sin\varphi$ kabi almashtirib topamiz:

$$\iint_{(D)}\left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy=\int_0^{2\pi}\left(\int_0^1(kc\sqrt{1-\rho^2}-\rho^2)ab\rho d\rho\right)d\varphi=$$

$$=ab\int_0^{2\pi}\left(\int_0^1 kc\rho\sqrt{1-\rho^2}-\rho^3\right)d\rho d\varphi=2\pi ab\left[-\frac{kc}{2}\cdot\frac{(1-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{\rho^4}{4}\right]=2\pi ab\left(\frac{kc}{3}-\frac{1}{4}\right).$$

Demak,

$$\iint_{(S)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy=2\pi\cdot ab\left(\frac{kc}{3}-\frac{1}{4}\right).$$

20.15. Birinchi va ikkinchi tur sirt integrallari orasidagi bog‘lanish

(S) sirtda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar berilgan bo‘lsin. Unda ushbu

$$\iint_{(S)}P(x,y,z)dydz+Q(x,y,z)dzdx+R(x,y,z)dxdy=$$

$$=\iint_{(S)}[P(x,y,z)\cos\alpha+Q(x,y,z)\cos\beta+R(x,y,z)\cos\gamma]ds$$

formula o‘rinli bo‘ladi.

20.16. Stoks formulasi

Stoks formulasi sirt bo‘yicha olingan integral bilan shu sirtning chegarasi bo‘yicha olingan egri chiziqli integralni bog‘lovchi formuladir.

Fazoda ikki tomoni silliq (S) sirt berilgan bo‘lib, uning chegarasi $\partial(S)$ esa bo‘lakli-silliq egri chiziqdan iborat bo‘lsin. (S) sirtda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar aniqlangan. Bu funksiyalar (S) da uzluksiz hamda barcha argumentlari bo‘yicha uzluksiz xususiy hosilalarga ega bo‘lsin. U holda ushbu

$$\oint_{\partial(S)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$\iint_{(S)} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy + \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right]dydz + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right]dzdx$$

formula o‘rinli bo‘ladi. Bu ***Stoks formulasi*** deyiladi.

Birinchi va ikkinchi tur sirt integrallarini o‘zaro bog‘lovchi formuladan foydalanib, Stoks formulasini quyidagicha ham yozish mumkin:

$$\oint_{\partial(S)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$\iint_{(S)} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right]ds \qquad (1)$$

Misol. Ushbu

$$\oint_{k} (y+z)dx + (z+x)dy + (x+y)dz$$

integralni hisoblang, bunda k yopiq chiziq bo‘lib,

$$x = a\sin^2 t,\ y = 2a\sin t\cos t,\ z = a\cos^2 t \quad (0 \le t \le 2\pi)$$

ellipsdan iborat.

Bu integralni Stoks formulasidan foydalanib hisoblaymiz.

$$P = y + z,\quad Q = z + x,\quad R = x + y$$

bo‘lib,

$$\frac{\partial P}{\partial y} = 1, \frac{\partial P}{\partial z} = 1, \frac{\partial Q}{\partial z} = 1, \frac{\partial Q}{\partial x} = 1, \frac{\partial R}{\partial x} = 1, \frac{\partial R}{\partial y} = 1$$

bo‘ladi. (3) formulaga binoan

$$\oint_{k} (y+z)dx + (z+x)dy + (x+y)dz =$$

$$\iint_{(S)} [(1-1)\cos\alpha + (1-1)\cos\beta + (1-1)\cos\gamma]ds = 0$$

bo‘ladi.

20.17. Ostrogradskiy formulasi

Fazoda, pastdan $z = \varphi_1(x,y)$ tenglama bilan aniqlangan silliq (S_1), yuqoridan $z = \varphi_2(x,y)$ $(\varphi_1(x,y) \le \varphi_2(x,y))$ tenglama bilan aniqlangan silliq (S_2) sirtlar bilan, yon tomonlaridan esa OZ o‘qiga parallel bo‘lgan silindrik (S_3) sirt bilan chegaralangan (V) sohani qaraylik. (V) da $R(x,y,z)$ funksiya aniqlangan va uzluksiz bo‘lib, (V) da uzluksiz

$$\frac{\partial R(x,y,z)}{\partial z}$$

xususiy hosilaga ega bo‘lsin. U holda

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z}\,\text{dxdydz}=\iint\limits_{(S)} R(x,y,z)dxdy \qquad (1)$$

bo'ladi, bunda (S) sirt (V) jismni o'rab turuvchi sirt.

Xuddi shunga o'xshash (V) jism hamda $P(x,y,z)$, $Q(x,y,z)$ funksiyalar tegishli shartlarni qanoatlantirganda

$$\iiint\limits_{(V)} \frac{\partial Q(x,y,z)}{\partial y}dxdydz = \iint\limits_{(S)} P(x,y,z)dxdz, \quad (2)$$

$$\iiint\limits_{(V)} \frac{\partial Q(x,y,z)}{\partial y}dxdydz = \iint\limits_{(S)} Q(x,y,z)dxdz \quad (3)$$

formulalar o'rinli bo'ladi

Aytaylik, (V) jism yuqoridagi (1), (2), (3) formulalarni o'rinli bo'lishida qo'yilgan shartlarni bajarsin.

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz = \iint\limits_{(S)} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$$

bo'ladi. Bu ***Ostrogradskiy formulasi*** deyiladi.

Birinchi va ikkinchi tur sirt integrallarini o'zaro bog'lovchi formuladan foydalanib, Ostrogradskiy formulasini quyidagicha ham yozish mumkin:

$$\iiint\limits_{(V)} \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz = \iint\limits_{(S)} [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]ds\,. \qquad (4)$$

Misol. Fazodagi (V) jismning hajmi

$$V = \frac{1}{3}\iint\limits_{(S)} (x\cos\alpha + y\cos\beta + z\cos\gamma)ds$$

bo'lishini isbotlang, bunda (S) sirt (V) jismni o'rab turgan sirt, $\cos\alpha, \cos\beta, \cos\gamma$ lar (S) sirt tashqi normalining yo'naltiruvchi kosinuslari.

Ostrogradskiy formulasining (4) ko'rinishidan foydalanib topamiz:

$$\iint\limits_{(S)} (x\cos\alpha + y\cos\beta + z\cos\gamma)ds = \iiint\limits_{(V)} \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)dxdydz = 3\iiint\limits_{(V)} dxdydz$$

$$\iiint\limits_{(V)} dxdydz = V$$

formulani e'tiborga olib, yuqoridagi tenglikdan

$$V = \frac{1}{3}\iint\limits_{(S)} (x\cos\alpha + y\cos\beta + z\cos\gamma)ds$$

bo'lishini topamiz.

M15. Egri chiziqli integrallarga doir mashqlar

1. Quyidagi birinchi tur egri chiziqli integrallarni hisoblang.

1) $\int\limits_{r}(x+y)dS$, bu yerda r -uchlari $O(0;\,0)$, $A(1;0)$, $B(0;1)$ nuqtalarda yotuvchi uchburchak konturidan iborat.

2) $\int\limits_{r}xydS$, bu yerda r - $x=0$, $y=0$, $x=4$, $y=2$ to'g'ri chiziqlar bilan chegaralangan to'rtburchak konturidan iborat.

3) $\int\limits_{r}(x^2+y^2)^5dS$, bu yerda r - $x=a\cos t$, $y=a\sin t$ aylanadan iborat.

4) $\int\limits_{\breve{AB}}(x+y)dS$, bunda AB chiziq tekislikni (0; 2) va (2; 0) nuqtalarini birlashtiruvchi to'g'ri chiziq kesmasi.

5) $\int\limits_{\breve{AB}}\frac{1}{\sqrt{x^2+y^2+4}}dS$, bunda $\breve{AB}$ tekislikning (0; 0) va (1; 2) nuqtalarini birlashtiruvchi to'g'ri chiziq kesmasi.

6) $\int\limits_{r}\sqrt{x^2+y^2}\,dS$, bu yerda r $x^2+y^2=ax$ aylanadan iborat.

7) $\int\limits_{\breve{AB}}\frac{1}{x+y}dS$, bunda $\breve{AB}$ ushbu $y=x+2$ to'g'ri chiziqni (2; 4) va (1; 3) nuqtalari orasidagi qismi.

8) $\int\limits_{\breve{AB}}ydS$, bunda $\breve{AB}$ quyidagi $y^2=2x$ parabolaning (0; 0) va (1; $\sqrt{2}$) nuqtalari orasidagi yoyi.

2. Quyidagi ikkinchi tur egri chiziqli integralni hisoblang.

1) $\int\limits_{\breve{AB}}(x^2-2xy)dx+(y^2-2xy)dy$, bu yerda $\breve{AB}$ $y=x^2$ $(0\le x\le 1)$ parabola.

2) $\int\limits_{r}xdy$, bu yerda r $\frac{x}{a}+\frac{y}{b}=1$ to'g'ri chiziqning Ox o'qi bilan kesishish nuqtasidan Oy o'qi bilan kesishish nuqtasigacha bo'lgan kesmasi.

3) $\int\limits_{r}xdx+xydy$, bu yerda r soat strelkasining harakatiga teskari yo'nalgan $x^2+y^2=2x$ yuqori yarim aylana.

4) $\int\limits_{r}\frac{xy(ydx-xdy)}{x^2+y^2}$, bu yerda r $\rho=a\sqrt{\cos 2\varphi}$ lemniskataning soat strelkasining harakatiga teskari yo'nalgan o'ng yaprog'i.

5) $\int\limits_{L}(x^2-2xy)dx+(y^2-2xy)dy$, $L=\{(x,y):y=x^2,\,-1\le x\le 1\}$

6) $\int\limits_{L}(x^2+y^2)dx+(x^2-y^2)dy$, $L=\{(x,y):y=1-|1-x|,\,0\le x\le 2\}$

7) $\int\limits_{L}(x+y)dx+(x-y)dy$, $L=\left\{(x,y):\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$

8) $\int\limits_{L}(2a+y)dx+xdy$, $L=\{(x,y):x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi\}$

Testlar

1) r: $A(1;\,0)$ va $B(0;\,1)$ nuqtalarni tutashtiruvchi kesma bo'lsa, $\int\limits_r ydx - xdy$ integralni hisoblang.

A) 1 B) 2 C) 3 D) 1,2 E) 1,3

2) r: $A(1;\,0)$ va $B(0;\,1)$ nuqtalarni tutashtiruvchi aylana yoyi bo'lsa, $\frac{2}{\pi}\int\limits_r ydx - xdy$ integralni hisoblang.

A) 1 B) 2 C) 3 D) 1,2 E) 1,3

3) $P(x,y)$, $Q(x,\,y)$ funksiyalari bir bog'lamli $\Omega \in R^2$ sohada uzluksiz differensiallanuvchi, $G \subset R$ sodda, bo'lakli silliq egri chiziq $G \subset \Omega$ sohaning chegarasi bo'lsa, $\int\limits_G Pdx - Qdy$ ga teng ifodani toping.

1) $\iint\limits_G \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)dxdy$; 2) $\iint\limits_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$; 3) $\iint\limits_G \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)$

A) 1 B) 2 C) 3 D) 1,2 E) 1,3

5. Quyidagi birinchi tur sirt integrallarini hisoblang.

1) $\iint\limits_S \left(x^2 + y^2\right)dS$, bunda S $x^2 + y^2 + z^2 = a^2$ sfera.

2) $\iint\limits_S \sqrt{x^2 + y^2}\,dS$, bunda S $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0$ $(0 \le z \le b)$ konus yon sirti.

6. Quyidagi ikkinchi tur sirt integrallarini hisoblang.

1) $\iint\limits_S yzdydz + xzdzdx + xydxdy$, bunda S $x = 0$, $y = 0$, $z = 0$, $x + y + z = a$ tekisliklar bilan chegaralangan tetraedrning tashqi sirti.

2) $\iint\limits_S zdxdy$, bunda S $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidaning tashqi sirti.

3) $\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy$, bunda S $x^2 + y^2 + z^2 = a^2$ $(z \ge 0)$ sfera yarim sirti.

4) $I_1 = \iint\limits_S x^2dydz$, $I_1 = \iint\limits_S y^2dzdx$ integrallarni hisoblang, bunda (S) sirt $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$ sferaning tashqi tomoni.

5) $\iint\limits_S zdxdy + ydxdz + zdydz$ integralni hisoblang, bunda (S) sirt $\{(x,y,z) \in R^3 : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ kubning tashqi sirti.

6) Ushbu $\oint\limits_K e^x dx + z(x^2 + y^2)^{\frac{3}{2}}dy + yz^3dz$ integralni Stoks formulasidan foydalanib hisoblang, bunda K egri chiziq $z = \sqrt{x^2 + y^2}$ sirtning $x = 0$,

$x=2$, $y=0$, $y=1$ tekisliklar bilan kesishgan chiziqlardan tashkil topgan yopiq chiziqdir.

7) $\oint\limits_K (y+z)dx+(z+x)dy+(x+y)dz$ integralni Stoks formulasidan foydalanib hisoblang, bunda K yopiq chiziq $x=a\sin^2 t$, $y=2a\sin t\cos t$, $z=a\cos^2 t$ ($0\le t\le 2\pi$) ellipsdan iborat.

8) Ushbu $\iint\limits_{(S)} x^2dydz+y^2dzdx+z^2dxdy$ integralni Ostrogradiyskiy formulasidan foydalanib hisoblang, bunda (S) sirt $\{(x,y,z)\in R^3 : 0\le x\le a,\ 0\le y\le a,\ 0\le z\le a\}$ kubning tashqi tomoni.

9) Stoks formulasidan foydalanib quyidagi egri chiziqli integrallarni sirt integrallari orqali ifodalang.

9.1. $\oint\limits_K ydx+zdy+xdz$; 9.2. $\oint\limits_K x^2y^3dx+dy+dz$

9.3. $\oint\limits_K (y^2+z^2)dx+(x^2+z^2)dy+(x^2+y^2)dz$

10) Ostrogradiyskiy formulasidan foydalanib, sirt integralini hisoblang, $\iint\limits_{(S)} xdydz+ydzdx+zdxdy$, bunda (S) sirt $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidaning tashqi tomoni.

21-§. FURYE QATORLARI

21.1. Furye qatori tushunchasi

$f(x)$ funksiya $[-\pi,\pi]$ da berilgan va shu oraliqda integrallanuvchi bo‘lsin. Ravshanki, $f(x)\cos nx$, $f(x)\sin nx$ $(n=1,2,3,...)$ funksiyalar ham $[-\pi,\pi]$ da integrallanuvchi bo‘ladi. Quyidagi belgilashlarni kiritamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \quad (n=1,2,3\cdots), \tag{1}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx, \quad (n=1,2,3,\cdots).$$

Ta’rif. Ushbu

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{2}$$

funksional qator $[-\pi,\pi]$da berilgan $f(x)$ funksiyaning ***Furye qatori*** deyiladi. $a_0, a_1, b_1, \cdots a_n, b_n, \cdots$ sonlar $f(x)$ funksiyaning Furye koeffitsiyentlari.

(2) qator $f(x)$ funksiyaning Furye qatori bo‘lishi quyidagicha yoziladi:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Agar $f(x)$ juft funksiya bo‘lsa, u holda uning Furye koeffitsiyentlari

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx, \quad (n = 1,2,3,\ldots)$$

$$b_n = 0 \quad (n = 1,2,3,\ldots)$$

bo‘lib, Furye qatori esa

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

bo‘ladi.

Agar $f(x)$ toq funksiya bo‘lsa, u holda uning Furye koeffitsiyentlari

$$a_n = 0, (n = 1,2,3,\cdots)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \quad (n = 1,2,3,\cdots)$$

bo‘lib, Furye qatori esa

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

bo‘ladi.

21.2. $[-l,l]$ oraliqda berilgan funksiyaning Furye qatori

$f(x)$ funksiya $[-l,l]$ da $(l > 0)$ berilgan va shu oraliqda integrallanuvchi bo‘lsin. Quyidagi belgilashlarni kiritamiz:

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x dx \quad (n = 1,2,\ldots),$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x dx \quad (n = 1,2,\ldots).$$

Ta’rif. Ushbu

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)$$

funksional qator $[-l,l]$ da berilgan $f(x)$ funksiyaning ***Furye qatori*** deyiladi. $a_0, a_1, b_1, \cdots a_n, b_n, \cdots$ sonlar Furye koeffitsiyentlari.

(2) qator $f(x)$ funksiyaning Furye qatori bo‘lishi quyidagicha yoziladi:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

Misol. Ushbu

$$f(x) = x^2 \qquad (-\pi \le x \le \pi)$$

juft funksiyaning Furye qatorini yozing.

Yuqoridagi (1) formulalardan hamda funksiyaning juftligidan foydalanib, berilgan funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx \frac{2}{3}\pi^2 \text{,}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi}x^2\frac{\sin nx}{n}\Bigg|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx dx =$$

$$= -\frac{4}{n\pi}\left[\left(-x\frac{\cos nx}{n}\right)\Bigg|_0^{x} + \int_0^{\pi}\cos nx dx\right] == (-1)^n\frac{4}{n^2} \quad (n = 1,2,3,\ldots)$$

Demak, $f(x) = x^2$ funksiyaning Furye qatori

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx$$

bo‘ladi.

21.3. Furye qatorining yaqinlashuvchiligi

Teorema. 2π davrli $f(x)$ funksiya $[-\pi,\pi]$ da bo‘lakli-differensiallanuvchi bo‘lsa, u holda bu funksiyaning Furye qatori

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

$[-\pi,\pi]$ da yaqinlashuvchi bo‘lib, $x \in (-\pi,\pi)$ da

$$\frac{f(x+0)+f(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

bo‘ladi.

$x = \pm\pi$ bo‘lganda $f(x)$ funksiya Furye qatorining yig‘indisi

$$\frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right]$$

ga teng bo‘ladi.

Teorema. Agar 2π davrli $f(x)$ funksiya $[-\pi,\pi]$ da uzluksiz, bo‘lakli-differensiallanuvchi va $f(-\pi) = f(\pi)$ bo‘lsa, bu funksiyaning Furye qatori $[-\pi,\pi]$ da yaqinlashuvchi bo‘lib,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

bo'ladi.

Misol. Ushbu

$$f(x)=\cos ax \qquad\qquad (0<a<1)$$

funksiyani Furye qatoriga yoying.

Bu funksiyaning Furye koeffitsiyentlarini hisoblaymiz:

$$a_0=\frac{2}{\pi}\int_0^{\pi}\cos ax dx=2\frac{\sin a\pi}{a\pi},$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\cos ax\cdot\cos nx dx=\frac{1}{\pi}\int_0^{\pi}[\cos(a+n)x+\cos(a-n)x dx=(-1)^n\cdot\frac{2a}{a^2-\pi^2}\cdot\frac{\sin a\pi}{\pi}$$

$$(n=1,2,3\cdots)$$

$$b_n=0 \quad (n=1,2,3\cdots)$$

Demak, berilgan funksiyaning Furye qatori

$$\cos ax\sim\frac{\sin a\pi}{a\pi}+\frac{2a\sin a\pi}{a\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx$$

bo'ladi. Qaralayotgan funksiya yuqoridagi teoremaning shartlarini bajaradi. Shuning uchun $f(x)=\cos ax$ funksiya Furye qatoriga yoyiladi:

$$\cos ax=\frac{\sin a\pi}{a\pi}+\frac{2a\sin a\pi}{a\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx.$$

Furye qatorlariga doir misollar

1. Quyidagi funksiyalarni Furye qatorlariga yoying.

1) $f(x)=e^x \; (-1\le x\le 1)$;
2) $f(x)=\sin^2 x$;
3) $f(x)=\cos^3 x$;
4) $f(x)=2x$, $x=(0;1)$
5) $f(x)=e^{ax}$, $x=(-h;h)$;
6) $f(x)=\cos 2x$, $x=(-\pi;\pi)$, $x\in Z$
7) $f(x)=\sin 2x$, $x=(-\pi;\pi)$, $x\in Z$

2. Furullani formulasidan foydalanib integrallarni hisoblang.

2.1. $\int_0^{\infty}\frac{\cos^2 ax-\cos^2 bx}{x}dx$;
2.2. $\int_0^{\infty}\frac{e^{-ax^2}-e^{-bx^2}}{x}dx$

3. Dirixle integralidan foydalanib integralni hisoblang.

3.1. $\int_0^{\infty}\frac{1-\cos ax}{x^2}dx$;
3.2. $\int_0^{\infty}\frac{x-\sin x}{x^3}dx$

ELEMENTAR MATEMATIKA FORMULALARI

Natural sonlar

Natural sonlar deb sanash uchun ishlatiladigan sonlarga aytiladi va quyidagicha belgilanadi:

N={1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,…}.

N – natural sonlar 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 raqamlaridan tuziladi.

Natural sonlarni quyidagicha yozish mumkin: $\overline{ab}=10a+b$, $\overline{abc}=100a+10b+c$, $\overline{abcde}=10000a+1000b+100c+10d+e$ va hokazo.

NATURAL SONLARGA BO‘LINISH ALOMATLARI

Ikkiga bo‘linadigan sonlar **juft sonlar** deyiladi va ular 0, 2, 4, 6, 8 – raqamlari bilan tugaydi.

Ikkiga bo‘linmaydigan sonlar **toq sonlar** deyiladi va ular 1, 3, 5, 7, 9 – raqamlari bilan tugaydi.

- **2 ga:** hamma juft sonlar 2 ga bo‘linadi;
- **3 ga:** raqamlar yig‘indisi 3 ga bo‘linadigan sonlar 3 ga bo‘linadi;
- **9 ga:** raqamlar yig‘indisi 9 ga bo‘linadigan sonlar 9 ga bo‘linadi;
- **4 ga:** oxirgi 2 ta raqami 0 bo‘lgan, yoki oxirgi ikki raqamidan tuzilgan son 4 ga bo‘linadigan sonlar 4 ga bo‘linadi;
- **25 ga:** oxirgi 2 ta raqami 0 bo‘lgan, yoki oxirgi ikki raqamidan tuzilgan son 25 ga bo‘linadigan sonlar, 25 ga bo‘linadi;
- **8 ga:** oxirgi 3 ta raqami 0 bo‘lgan, yoki oxirgi uchta raqamidan tuzilgan son 8 ga bo‘linadigan sonlar 8 ga bo‘linadi;
- **125 ga:** oxirgi 3 ta raqami 0 bo‘lgan, yoki oxirgi uchta raqamidan tuzilgan son 125 ga bo‘linadigan sonlar 125 ga bo‘linadi;
- **2^n ga:** oxirgi n ta raqami 0 bo‘lgan, yoki oxirgi n ta raqamidan tuzilgan son 2^n ga bo‘linadigan sonlar 2^n ga bo‘linadi;
- **5^n ga:** oxirgi n ta raqami 0 bo‘lgan, yoki oxirgi n ta raqamidan tuzilgan son 5^n ga bo‘linadigan sonlar, 5^n ga bo‘linadi.
- **6 ga:** ham 2 ga, ham 3 ga bo‘linadigan sonlar 6 ga bo‘linadi;
- **12 ga:** berilgan son 4 ga ham, 3 ga ham bo‘linsa, u vaqtda ushbu son 12 ga bo‘linadi;
- **15 ga:** berilgan son 3 ga ham, 5 ga ham bo‘linsa, u vaqtda ushbu son 15 ga bo‘linadi;
- $m\cdot n$ **ga:** berilgan *a* soni *m* ga ham, *n* ga ham (EKUB(*m*,*n*)=1) bo‘linsa, u vaqtda *a* soni $m\cdot n$ ga bo‘linadi;
- **5 ga:** oxirgi raqami 0 yoki 5 bo‘lgan sonlar 5 ga bo‘linadi;
- **10 ga:** oxirgi raqami 0 bo‘lgan sonlar 10 ga bo‘linadi;
- **36 ga:** oxirgi 2 ta raqami 0 bo‘lgan, raqamlari yig‘indisi 9 ga va oxirgi ikki raqamidan tuzilgan son 4 ga bo‘linadigan sonlar 36 ga bo‘linadi;

- **45 ga:** raqamlari yig'indisi 9 ga bo'linadigan va oxirgi raqami 0 yoki 5 bo'lgan sonlar 45 ga bo'linadi;
- **60 ga:** oxirgi raqami 0 bo'lgan va ham 2 ga, ham 3 ga bo'linadigan sonlar 60 ga bo'linadi;
- **7 ga:** oxirgi raqamining ikkilanganini qolgan raqamlardan tuzilgan sondan ayirganda 7 ga karrali son hosil bo'lsa, yoki 0 chiqsa, berilgan son 7 ga bo'linadi: Masalan: 91: $9-2\cdot1=7$, 1127: $112-2\cdot7=98=7\cdot14$;
- **11 ga:** toq o'rindagi raqamlar yig'indisi bilan juft o'rindagi raqamlar yig'indisining ayirmasi 0 yoki 11 ga karrali son bo'lsa ushbu son 11 ga bo'linadi: Masalan: 9873424 (9+7+4+4) – (8+3+2)=11;

Qolgan sonlarga bo'linish belgilarini ularni tub ko'paytuvchilarga ajratish yo'li bilan aniqlash mumkin.

BUTUN SONLAR

Natural sonlarga qarama-qarshi sonlar manfiy butun sonlar deyiladi.

Natural sonlar, nol soni va manfiy butun sonlar to'plamlarining birlashmasi **butun sonlar** to'plami deyiladi va u quyidagicha belgilanadi:

$$Z=\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

Bu to'plamdagi 0 soni ixtiyoriy $n \in N$ uchun $n+0=n$ va $n+(-n)=0$ tengliklarni qanoatlantiradi. Bu yerda n va $-n$ o'zaro qarama-qarshi sonlar deb ataladi.

RATSIONAL SONLAR

$\frac{m}{n}$, $(m \in Z,\ n \in N)$ ko'rinishidagi songa **ratsional son** deyiladi. Barcha ratsional sonlar to'plami quyidagicha belgilanadi:

$$Q=\left\{x /\quad x=\frac{m}{n},\ m \in Z,\ n \in N\right\}.$$

IRRATSIONAL SONLAR

Irratsional sonlar deb $\frac{m}{n}$, $(m \in Z,\ n \in N)$ ko'rinishida yozib bo'maydigan sonlarga aytiladi. Masalan, $\pi, \sqrt{2}, \sqrt{5}, \ldots$. Shunday sonlar to'plami I bilan belgilanadi.

Irratsional va ratsional sonlar to'plamlari birlashmasi **haqiqiy sonlar** to'plami deyiladi va R bilan belgilanadi, ya'ni $R=I \cup Q$.

TUB VA MURAKKAB SONLAR

Agar natural son faqat 1 ga va o'zigagina bo'linsa, u **tub son** deyiladi. Masalan: 2,3,5,7,11,13,17,19, ...

Quyida 1000 gacha bo'lgan tub sonlar jadvali keltirilgan:

TUB SONLAR JADVALI (1000 GACHA)

2	61	149	239	347	443	563	659	773	887
3	67	151	241	349	449	569	661	787	907
5	71	157	251	353	457	571	673	797	911

7	73	163	257	359	461	577	677	809	919
11	79	167	263	367	463	587	683	811	929
13	83	173	269	373	467	593	991	821	937
17	89	179	271	379	479	599	701	823	941
19	97	181	277	383	487	601	709	827	947
23	101	191	281	389	491	607	719	829	953
29	103	193	283	397	499	613	727	839	967
31	107	197	293	401	503	617	733	853	971
37	109	199	307	409	509	619	739	857	977
41	113	211	311	419	521	631	743	859	983
43	127	223	313	421	523	641	751	863	991
47	131	227	317	431	541	643	757	877	997
53	137	229	331	433	547	647	761	881	
59	139	233	337	439	557	653	769	883	

Agar natural sonning 1 va o‘zidan boshqa bo‘luvchilari mavjud bo‘lsa, u **murakkab son** deyiladi.

1 soni na tub na murakkab son deb qabul qilingan.

Har qanday sonni bir usulda tub sonlar ko‘paytmasi ko‘rinishida ifodalash mumkin.

BERILGAN SONNING BO‘LUVCHILARI SONINI VA YIG‘INDISINI TOPISH

Har qanday sonning bo‘luvchilari sonini topish uchun shu son tub ko‘paytuvchilarga ajratiladi va ko‘paytmada qatnashgan har bir hadning darajasiga birni qo‘shib, natijalar ko‘paytiriladi, ya’ni $a = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_m^{n_m}$ bo‘lsa,

$$A(a)=(n_1+1)(n_2+1)\ldots(n_m+1),$$

bu yerda $p_1, p_2, \ldots, p_m$ - tub sonlar.

“*a*” soning bo‘luvchilari yig‘indisini topish uchun quyidagi formuladan foydalanamiz:

$$S(a) = \frac{p_1^{n_1+1}-1}{p_1-1} \cdot \frac{p_2^{n_2+1}-1}{p_2-1} \cdot \ldots \cdot \frac{p_{m,}^{n_m+1}-1}{p_m-1}$$

Masalan: $500=2^2 \cdot 5^3$, $600=2^3 \cdot 3^1 \cdot 5^2$

A(500)=(2+1)(3+1)=12, A(600)=(3+1)(1+1)(2+1)=24.

$$S(500) = \frac{2^3-1}{2-1} \cdot \frac{5^4-1}{5-1} = 7 \cdot 156 = 1092,$$

$$S(600) = \frac{2^4-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^3-1}{5-1} = 15 \cdot 124 = 1860.$$

«*a*» soning tub yoki murakkab son ekanligini aniqlash uchun, $\sqrt{a} \geq p$, (bu yerda p- $\sqrt{a}$ dan katta bo‘lmagan eng katta tub son) p - tub songacha bo‘lgan sonlarga bo‘lib ko‘rish kerak.

Misol: a=1601 sonining tub son ekanligini aniqlash uchun, $\sqrt{1601}=40{,}012...$, demak 1601 ning tub yoki tub emasligini bilish uchun 2,3,5,7,…,37 tub sonlarga bo‘lib ko‘rish kifoya.

NATURAL SONLAR USTIDA ARIFMETIK AMALLAR

1. $a+b=c$ – qo‘shish amali bajariladi: 3+4=7.
2. $a\cdot b=n$ - ko‘paytirish amali bajariladi: $3\cdot4=12$.
3. a^m - darajaga ko‘tarish amali bajariladi: $3^2=3\cdot3=9$.

Ayirish va bo‘lish amallari hamma vaqt ham bajarilmaydi.

Masalan: 5-8=-3 natural son emas. $5:3=1\frac{2}{3}$. $1\frac{2}{3}$ natural son emas.

NATURAL SONLARNI QO‘SHISH VA KO‘PAYTIRISH AMALLARINING XOSSALARI

1. $a+b=b+a$ - qo‘shishning o‘rin almashtirish (kommutativlik) xossasi.
2. $(a+b)+c=a+(b+c)$ - qo‘shishning guruhlash (assosiativlik) xossasi.
3. $a\cdot b=b\cdot a$ - ko‘paytirishning o‘rin almashtirish (kommutativlik) xossasi.
4. $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ – ko‘paytirishning guruhlash (assosiativlik) xossasi.
5. $a\cdot(b+c)=a\cdot b+a\cdot c$ - qo‘shish va ko`paytirishning taqsimot (distributivlik) xossasi.

ENG KATTA UMUMIY BO‘LUVCHI (EKUB)

Berilgan sonlar bo‘linadigan sonlarning eng kattasiga shu sonlarning EKUBi deb aytiladi va u quyidagicha topiladi.

1. Berilgan sonlar tub ko‘paytuvchilarga ajratiladi.
2. Har bir sonning tub ko‘paytuvchilarga yoyilmasida qatnashgan umumiy tub ko‘paytuvchilarning eng kichik darajalilari olinadi.
3. Yozib olingan sonlar o‘zaro ko‘paytirilib, EKUB topiladi.

ENG KICHIK UMUMIY KARRALI (EKUK)

Berilgan sonlarga bo‘linadigan sonlarning eng kichigiga shu sonlarning EKUKi deb aytiladi va quyidagicha topiladi.

1. Berilgan sonlar tub ko‘paytuvchilarga ajratiladi.
2. Har bir sonning tub ko‘paytuvchilarga yoyilmasiga qatnashgan barcha tub ko‘paytuvchilarning eng katta darajalilari olinadi.
3. Yozib olingan sonlar o‘zaro ko‘paytirilib, EKUB topiladi.

Masalan, EKUB(28,144) va EKUK(28,144) ni toping.

28	2	144	2	$28=2^2\cdot7$,
14	2	72	2	$144=2^4\cdot3^2$,
7	7	36	2	EKUB(28,144)=2^2=4,
1		18	2	EKUK(28,144)=$2^4\cdot3^2\cdot7$=1008.
		9	3	
		3	3	
		1		

EKUK va EKUB uchun quyidagi munosabat o‘rinli:

$$EKUK(a,b)\cdot EKUB(a,b)=a\cdot b.$$

Eng katta umumiy bo‘luvchisi 1 ga teng bo‘lgan sonlarga **o‘zaro tub** sonlar deyiladi.

O‘ZARO TESKARI SONLAR

Ko‘paytmasi birga teng bo‘lgan sonlarga **o‘zaro teskari sonlar** deb aytiladi. *a* ga teskari son $\frac{1}{a}$, chunki $a\cdot\frac{1}{a}=1$ $(a\neq 0)$.

O‘ZARO QARAMA-QARSHI SONLAR

Yig‘indisi nolga teng bo‘lgan sonlar **o‘zaro qarama-qarshi sonlar** deyiladi. *a* ga qarama-qarshi son: $-a$, chunki $a+(-a)=0$.

QOLDIQLI BO‘LISH

$a=pq+r$ $(0\le r<p)$, bu yerda *a*-bo‘linuvchi, *p*-bo‘luvchi, *q*-bo‘linma, *r*-qoldiq.

ODDIY KASRLAR

$\frac{m}{n},(m\in Z, n\in N)$ ko‘rinishidagi ifoda **oddiy kas**r deyiladi. m - kasrning surati, n - kasrning maxraji deyiladi.

Oddiy kasrlar ustida quyidagi amallar bajariladi:

Qo‘shish va ayirish amallari:

Bir xil maxrajli kasrlar uchun: $\frac{a}{b}\pm\frac{c}{b}=\frac{a\pm c}{b}$.

Har xil maxrajli kasrlar uchun: $\frac{a}{b}\pm\frac{c}{d}=\frac{da\pm bc}{bd}$ yoki

$$\frac{a}{b}\pm\frac{c}{d}=\frac{a\cdot\frac{EKUK(b,d)}{b}\pm c\cdot\frac{EKUK(b,d)}{d}}{EKUK(b,d)}.$$

Ko‘paytirish amali: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

Bo‘lish amali: $\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$, $1:\frac{c}{d}=1\cdot\frac{d}{c}=\frac{d}{c}$.

$$a:b\cdot c=\frac{ac}{b}=(ac):b, \qquad a\cdot b:c\cdot d=\frac{a\cdot b\cdot d}{c}=(a\cdot b\cdot d):c.$$

Kasrlarni taqqoslash: Agar musbat kasrlarning suratlari bir xil bo‘lsa, maxraji kattasi kichik bo‘ladi. Agar musbat kasrlarning maxraji bir xil bo‘lsa, surati kattasi katta bo‘ladi. Misol: $\frac{1}{2}>\frac{1}{10}$, $\frac{4}{87}<\frac{13}{87}$

$\frac{a}{b}$ *va* $\frac{c}{d}$ musbat kasrlar berilgan bo‘lsin. Agar $ad>bc$ bo‘lsa, u holda $\frac{a}{b}>\frac{c}{d}$ bo‘ladi. Agar $ad<bc$ bo‘lsa, u holda $\frac{a}{b}<\frac{c}{d}$ bo‘ladi.

ARALASH KASRNI NOTO‘G‘RI KASRGA AYLANTIRISH

$a\frac{c}{b}=\frac{ab+c}{b}$. Masalan: $5\frac{2}{3}=\frac{5\cdot 3+2}{3}=\frac{17}{3}$.

DAVRIY KASRLAR

Butun qismidan keyin darhol davr boshlanadigan kasrlarga **sof davriy kasrlar** deyiladi. Masalan: 0,(3); 1,(21); 10,(07).

Sof davriy o'nli kasrni oddiy kasrga aylantirish uchun suratida butun qismi bilan birinchi davrdan iborat sondan davrgacha bo'lgan sonning ayirmasi, maxrajida esa 9 ni davrda necha raqam bo'lsa, shuncha yozib, uni verguldan birinchi davrgacha nechta raqam bo'lsa shuncha nol bilan to'ldirish kerak:

$$a_1a_2..a_k,(b_1b_2..b_n)=\frac{\overline{a_1a_2..a_kb_1b_2..b_n}-\overline{a_1a_2..a_k}}{\underbrace{99..9}_{n}\underbrace{00..0}_{k}}$$

$$0,(n)=\frac{n}{9};\quad 0,(\overline{an})=\frac{\overline{an}}{99};\quad k,(n)=\frac{\overline{kn}-k}{9};\qquad k,(\overline{bn})=k\frac{\overline{bn}}{99}.$$

Kasr qismida bir yoki bir nechta raqamdan so'ng davr boshlanadigan kasrlar **aralash davriy kasrlar** deyiladi. Masalan: 0,1(25); 5,05(50).

Aralash davriy kasrlarni oddiy kasrga aylantirish uchun butun qismi butun qismiga yozilib, kasr qismining suratiga kasr qismiga qatnashgan barcha raqamlaridan tuzilgan sondan davrgacha qatnashgan raqamlaridan tuzilgan son ayirilib natija yoziladi, maxrajiga esa davrdagi raqamlar sonicha 9 yoziladi va davrga kirmagan kasr qismidagi raqamlar sonicha 0 yoziladi.

$0,b(n)=\frac{\overline{bn}-b}{90}$, $\quad 0,\overline{ab}(\overline{cd})=\frac{\overline{abcd}-\overline{ab}}{9900}$, $\ k,b(n)=k\frac{\overline{bn}-b}{90}$, yoki $k,b(n)=\frac{\overline{kbn}-\overline{kb}}{90}$,

$k,ab(\overline{cd})=\frac{\overline{kabcd}-\overline{kab}}{9900}$, $\ k,m(\overline{bn})=\frac{\overline{kmbn}-\overline{km}}{990}$, yoki $k,m(bn)=k\frac{\overline{mbn}-m}{990}$.

Misollar: $0,(6)=\frac{6}{9}=\frac{2}{3}=\frac{12}{18}$, $0,1(23)=\frac{123-1}{990}=\frac{122}{990}=\frac{61}{495}$,

$1{,}2(35)=\frac{1235-12}{990}=\frac{1223}{990}$, yoki $1{,}2(35)=1\frac{235-2}{990}=1\frac{233}{990}$.

ARIFMETIK AMALLARNING XOSSALARI

1. $a+b=c$ bo'lsa, $c-b=a$ deyiladi;
2. $a\cdot b=c$ bo'lsa, $c:b=a$, $(b\neq 0)$, deyiladi;
3. $-(a+b-c)=-a-b+c$;
4. $m(a+b-c)=ma+mb-mc$;
5. $(a+b-c):m=a:m+b:m-c:m$;
6. $(a\cdot b)\cdot m=(a\cdot m)\cdot b=a\cdot(b\cdot m)$;
7. $(a\cdot b):m=(a:m)\cdot b=a\cdot(b:m)$;
8. $a+0=a-0=a$;
9. $a\cdot 1=a$;
10. $a:a=1, \quad (a\neq 0)$; 11. $0:a=0, \ (a\neq 0)$.

BA'ZI BIR YIG'INDILAR VA TENGSIZLIKLAR

1) $1+2+3+\ldots+n=\dfrac{n(n+1)}{2};$

2) $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6};$

3) $1^3+2^3+3^3+\ldots+n^3=\dfrac{n^2(n+1)^2}{4};$

4) $1^2+3^2+\ldots+(2n-1)^2=\dfrac{n(2n-1)(2n+1)}{3};$

5) $1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}\cdot n^2=(-1)^{n-1}\cdot\dfrac{n(n+1)}{2};$

6) $1+3+5+\ldots+(2n-1)=n^2;$ 7) $2+4+6+\ldots+2n=n(n+1);$

8) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1};$

9) $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\ldots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{(2n+1)};$

10) $2^n>n^2, n\geq 5;$

11) $\dfrac{1}{a(a+1)}+\dfrac{1}{(a+1)(a+2)}+\dfrac{1}{(a+n-1)(a+n)}=\dfrac{n}{a(a+n)}, bunda \quad a>0;$

12) $\dfrac{1}{(n+1)}+\dfrac{1}{(n+2)}+\ldots+\dfrac{1}{2n}>\dfrac{1}{2};$ 13) $\dfrac{1}{4}+\dfrac{1}{9}+\ldots+\dfrac{1}{n^2}<1;$

14) $\dfrac{1}{\sqrt{1}}+\dfrac{1}{\sqrt{2}}+\ldots+\dfrac{1}{\sqrt{n}}>\sqrt{n};$

15) Ixtiyoriy musbat haqiqiy $a_1, a_2, \ldots, a_n$ sonlar uchun

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}\geq n;$$

16) $\left(\dfrac{n}{3}\right)^n<n!<\left(\dfrac{n+1}{2}\right)^n, \ n\geq 3.$

PROPORSIYA

Ikki nisbatning tengligiga **proporsiya** deyiladi. $a:b=c:d$ bunda a,b,c,d lar noldan farqli sonlar. a va d lar proporsiyaning chetki hadlari, b va c lar proporsiyaning o'rta (ichki) hadlari deyiladi.

Proporsiya uchun quyidagi tasdiqlar o'rinli:

1. Proporsiyaning chetki hadlari ko'paytmasi o'rta hadlari ko'paytmasiga teng.
2. Proporsiyaning chetki (o'rta) hadlarini o'rinlarini almashtirish mumkin: $d:c=b:a$.
3. Proporsiyaning har bir hadini biror $k\neq 0$ songa ko'paytirish yoki bo'lish mumkin.
4. Proporsiyada $\dfrac{a}{b}=\dfrac{c}{d}$, $a=\dfrac{c\cdot b}{d}$, $b=\dfrac{a\cdot d}{c}$, $c=\dfrac{a\cdot d}{b}$, $d=\dfrac{b\cdot c}{a}$ tengliklar o'rinli.

5. $\dfrac{a}{b}=\dfrac{c}{d}$ bo‘lsa, quyidagi tengliklar o‘rinli: $\dfrac{a}{b}=\dfrac{m_1\cdot a+m_2\cdot c}{m_1\cdot b+m_2\cdot d}$,

xususiy hollarda $\dfrac{a+b}{b}=\dfrac{c+d}{d}$, $\dfrac{a-b}{b}=\dfrac{c-d}{d}$, $\dfrac{a}{a+b}=\dfrac{c}{c+d}$,

$\dfrac{a}{a-b}=\dfrac{c}{c-d}$, $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$ bo‘ladi.

Agar $ac=b^2$ bo‘lsa, $\dfrac{a^2+b^2}{b^2+c^2}=\dfrac{a}{c}$ bo‘ladi.

6. a sonini m:n:k nisbatda to‘g‘ri proportsional bo‘laklarga ajratish uchun quyidagi formulalardan foydalanamiz:

$$\frac{a}{m+n+k}\cdot m;\qquad \frac{a}{m+n+k}\cdot n;\qquad \frac{a}{m+n+k}\cdot k.$$

Teskari proportsional bo‘laklarga ajratishda esa quyidagi formulalardan foydalanamiz:

$$\frac{a}{\frac{1}{m}+\frac{1}{n}+\frac{1}{k}}\cdot\frac{1}{m};\qquad \frac{a}{\frac{1}{m}+\frac{1}{n}+\frac{1}{k}}\cdot\frac{1}{n};\qquad \frac{a}{\frac{1}{m}+\frac{1}{n}+\frac{1}{k}}\cdot\frac{1}{k}.$$

Agar $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\ldots=\dfrac{a_n}{b_n}$ bo‘lsa, u holda quyidagilar o‘rinli:

1) $\dfrac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}=\dfrac{a_1}{b_1}$ 2) $\dfrac{a_1m_1+a_2m_2+\ldots+a_nm_n}{b_1m_1+b_2m_2+\ldots+b_nm_n}=\dfrac{a_1}{b_1}$

bunda $m_1,\ldots,m_n$, - haqiqiy noldan farqli sonlar.

O‘RTA QIYMATLAR

Agar $x_1,x_2,\ldots,x_n$ berilgan sonlar bo‘lsa, o‘rta qiymatlar quyidagicha ta’riflanadi:

1) $x_1,x_2,\ldots,x_n$ sonlarning o‘rta **arifmetigi**: $A=\dfrac{x_1+x_2+\ldots+x_n}{n}$.

2) $x_1,x_2,\ldots,x_n$ musbat sonlarning o‘rta **geometrigi**: $G=\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}$.

3) $x_1,x_2,\ldots,x_n$ noldan farqli sonlarning o‘rta **garmonigi**: $H=\dfrac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}$.

4) $x_1,x_2,\ldots,x_n$ sonlarning o‘rta **proportsionali**: agar $\dfrac{a}{b}=\dfrac{b}{c}$ bo‘lsa, $b=\sqrt{ac}$.

5) $x_1,x_2,\ldots,x_n$ sonlarning o‘rta **kvadratigi**: $K=\sqrt{\dfrac{x_1^2+x_2^2+\ldots+x_n^2}{n}}$.

6) $x_1,x_2,\ldots,x_n$ sonlarning o‘rta **vazni**: $V=\dfrac{a_1x_1+a_2x_2+\ldots+a_nx_n}{a_1+a_2+\ldots+a_n}$.

7) Koshi teoremasi: $H\le G\le A$ yoki

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}\le\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}\le\frac{x_1+x_2+\ldots+x_n}{n}.$$

PROTSENT

Sonning 100 dan bir ulushi 1% (1%=0,01) **protsent** deyiladi.

1. a sonining p protsentini topish: $\begin{matrix} a - 100\% \\ x - p\% \end{matrix} \Rightarrow x = \dfrac{a \cdot p}{100}$.
2. p protsenti a ga teng bo‘lgan sonni topish: $\begin{matrix} x - 100\% \\ a - p\% \end{matrix} \Rightarrow x = \dfrac{a \cdot 100}{p}$.
3. a va b sonlarining protsent nisbatini topish: $\dfrac{a}{b} \cdot 100\%$.
4. Murakkab protsent: $A = a(1 \pm 0{,}01r)^n$.

Bunda n - qiymatning necha marta oshishi yoki kamayishi $(n \in N)$, a – dastlabki miqdorlar, r – (%) foiz, A- oxirgi qiymat.

QISQA KO‘PAYTIRISH FORMULALARI

1) $(a+b)^2 = a^2 + 2ab + b^2$. 2) $(a-b)^2 = a^2 - 2ab + b^2$.

3) $a^2 - b^2 = (a-b)(a+b)$.

4) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a+b) + b^3$.

5) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - 3ab(a-b) - b^3$.

6) $a^3 + b^3 = (a+b)(a^2 - ab + b^2) = (a+b)^3 - 3ab(a+b)$.

7) $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)^3 + 3ab(a-b)$.

Daraja va ildiz

a haqiqiy sonining 1 dan katta n **natural ko‘rsatkichli darajasi** deb har biri a ga teng bo‘lgan n ta ko‘paytuvchining ko‘paytmasiga aytiladi:

$$\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n\,\text{marta}}.$$

qisqacha a^n kabi belgilanadi, va a ning n ko‘rsatkichli darajasi deb o‘qiladi. Bu yerda a darajaning asosi, n shu darajaning ko‘rsatkichi deyiladi. Shuningdek ta’rifga ko‘ra, $a^1 = a$, $a^0 = 1$ $(a \neq 0)$, $a^{-k} = \dfrac{1}{a^k}$ $(a \neq 0)$ deb qabul qilingan.

a va b musbat sonlari uchun $b^n = a$ tenglik bajarilsa, b musbat soniga a musbat sonining n -darajali arifmetik ildizi deyiladi va $\sqrt[n]{a}$ yoki $a^{\frac{1}{n}}$ kabi belgilanadi, hamda $\sqrt[n]{\ }$ ni n -darajali radikal deb yuritiladi. $n = 2$ bo‘lganda 2-darajali ildizni (arifmetik) kvadrat ildiz deb aytiladi va oddiygina $\sqrt{a}$ kabi yoziladi.

Darajaning xossalari:

1) $a^n \cdot a^m = a^{n+m}$, 2) $(a \cdot b)^n = a^n \cdot b^n$,

3) $a^n : a^m = a^{n-m}$, 4) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$, $(b \neq 0)$

5) $(a^n)^m = a^{n \cdot m}$.

Ildizning xossalari:

1) $\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$. 2) $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$. 3) $\sqrt[m]{\sqrt[n]{a}}=\sqrt[nm]{a}$. 4) $\sqrt{a^2}=|a|$. 5) $a\sqrt[m]{b}=\sqrt[m]{ba^m}$.

6) $\sqrt[km]{a^{kn}}=\sqrt[m]{a^n}$ 6) $\sqrt[m]{a\sqrt[m]{a\sqrt[m]{a\sqrt[m]{a\ldots}}}}=\sqrt[m-1]{a}$. 7) $\sqrt[m]{a:\sqrt[m]{a:\sqrt[m]{a:\sqrt[m]{a\ldots}}}}=\sqrt[m+1]{a}$.

8) $\sqrt{a+\sqrt{a+\sqrt{a+\ldots}}}=\frac{1+\sqrt{1+4a}}{2}$. 9) $\sqrt{a-\sqrt{a-\sqrt{a-\ldots}}}=\frac{\sqrt{1+4a}-1}{2}$.

Ratsional ko‘rsatkichli daraja va uning xossalari:

$a^{\frac{n}{m}}=\sqrt[m]{a^n}$ tenglik ta’rif sifatida qabul qilingan.

1) $\sqrt[m]{a^n\cdot b^k\cdot c^l}=a^{\frac{n}{m}}\cdot b^{\frac{k}{m}}\cdot c^{\frac{l}{m}}$. 2) $a^{\frac{n}{m}}\cdot a^{\frac{p}{q}}=a^{\frac{n}{m}+\frac{p}{q}}$.

3) $a^{\frac{n}{m}}:a^{\frac{p}{q}}=a^{\frac{n}{m}-\frac{p}{q}}$. 4) $\left(a^{\frac{n}{m}}\right)^{\frac{p}{q}}=a^{\frac{n}{m}\cdot\frac{p}{q}}$.

Murakkab ildiz formulasi

$$\sqrt{A\pm\sqrt{B}}=\sqrt{\frac{A+\sqrt{A^2-B}}{2}}\pm\sqrt{\frac{A-\sqrt{A^2-B}}{2}}.$$

Masalan:

$$\sqrt{9-\sqrt{17}}=\sqrt{\frac{9+\sqrt{81-17}}{2}}-\sqrt{\frac{9-\sqrt{81-17}}{2}}==\frac{1}{\sqrt{2}}\left(\sqrt{9+8}-\sqrt{9-8}\right)=\frac{1}{\sqrt{2}}\left(\sqrt{17}-1\right).$$

KASRNING MAXRAJINI IRRATSIONALLIKDAN QUTQARISH

1) $\frac{b}{\sqrt[m]{a}}=\frac{b\sqrt[m]{a^{m-1}}}{\sqrt[m]{a}\sqrt[m]{a^{m-1}}}=\frac{b\sqrt[m]{a^{m-1}}}{a}$. Misol: $\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}}=\frac{3\sqrt{5}}{5}$.

2) $\frac{a}{\sqrt{x}\pm\sqrt{y}}=\frac{a\left(\sqrt{x}\mp\sqrt{y}\right)}{\left(\sqrt{x}\pm\sqrt{y}\right)\left(\sqrt{x}\mp\sqrt{y}\right)}=\frac{a\left(\sqrt{x}\mp\sqrt{y}\right)}{x-y}$.

Misol: $\frac{6}{\sqrt{7}-\sqrt{5}}=\frac{6\cdot\left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}-\sqrt{5}\right)\left(\sqrt{7}+\sqrt{5}\right)}=\frac{6\cdot\left(\sqrt{7}+\sqrt{5}\right)}{7-5}=3\cdot\left(\sqrt{7}+\sqrt{5}\right)$.

3) $\frac{b}{\sqrt{x}\mp y}=\frac{b\left(\sqrt{x}\pm y\right)}{\left(\sqrt{x}\mp y\right)\left(\sqrt{x}\pm y\right)}=\frac{b\left(\sqrt{x}\pm y\right)}{x-y^2}$.

Misol: $\frac{3}{\sqrt{11}-3}=\frac{3\cdot\left(\sqrt{11}+3\right)}{11-9}=\frac{3\cdot\left(\sqrt{11}+3\right)}{2}$.

4) $\frac{c}{\sqrt[3]{x}\pm\sqrt[3]{y}}=\frac{c\left(\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}\right)}{\left(\sqrt[3]{x}\pm\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}\right)}=\frac{c\left(\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}\right)}{x\pm y}$.

SONNING MODULI VA UNING XOSSALARI

$$|a|=\begin{cases}a, & a\geq 0 \text{ bo'lsa},\\ -a, & a<0 \text{ bo'lsa}.\end{cases}$$

1) $|a| \geq 0.$ 2) $|-a| = |a|.$ 3) $|a| \geq a.$ 4) $|a| \geq -a.$

5) $|a \cdot b| = |a| \cdot |b|.$ 6) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$ 7) $|a| = |b|$ bo'lsa $a = \pm b.$ 8) $|a \pm b| \leq |a| + |b|.$

9) $|a - b| \geq |a| - |b|.$ 10) $|a - b| \geq ||a| - |b||.$

11) $|x| > a,$ $a > 0$ bo'lsa $x > a, x < -a$ yoki $x \in (-\infty; -a) \cup (a; \infty)$ bo'ladi.

12) $|x| < a$ $(a > 0),$ $-a < x < a.$ 13) $|a|^2 = a^2.$

MANFIY SONLAR USTIDA AMALLAR

1. $-(-m) = m;$ 2. $m + (-n) = m - n;$ 3. $m - (-n) = m + n;$

4. $m(-n) = (-m)n = -mn;$ 5. $(-n)(-m) = nm;$

6. $(-m)(-n)(-p) = -mnp;$ 7. $m : (-n) = (-m) : n = -m : n;$

8. $(-m) : (-n) = m : n;$

9. $k \in N$ bo'lsa, $(-n)^k = \begin{cases} n^k, (k - \text{juft bo'lsa}); \\ -n^k, (k - \text{toq bo'lsa}). \end{cases}$

Misol: $(-3)^4=3^4$, $(-3)^5=-3^5$.

BIRHAD VA KO‘PHAD

Birhad deganda son, o‘zgaruvchi va o‘zgaruvchi darajalari ko‘paytmasidan tashkil topgan ifoda tushiniladi. Masalan: $3x^2, -5kxy$ va h.z.

Daraja ta’rifiga asosan eng ixcham tarzda yozilgan birhad **standart shakldа yozilgan birhad** deyiladi.

Standart shaklda yozilgan birhadning son ko‘paytuvchisi birhadning **koeffisiyenti** deyiladi.

Ko‘phad deb bir necha birhadlar yig‘indisidan tashkil topgan o‘zgaruvchili ifodaga aytiladi. Yig‘indida ishtirok etayotgan barcha birhadlar bir xil o‘zgaruvchining birhadlari bo‘lsa, ko‘phadni **bir o‘zgaruvhili ko‘phad** deyiladi, aks holda **ko‘p o‘zgaruvhili ko‘phad** deyiladi.

Bir o‘zgaruvhili ko‘phadda ishtirok etayotgan o‘zgaruvchining eng yuqori darajasi ko‘phadning **darajasi** deyiladi. Agar ko‘phadda ishtirok etayotgan bir xil darajali birhadlar ixchamlashtirilsa va darajaning o‘sishi (kamayishi) tartibida yozilsa, u quyidagi ko‘rinishni oladi: $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,\ a_n \neq 0,$ $\left(a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,\ a_0 \neq 0\right).$

Bu **n-darajali standart shakldagi ko‘phad** deyiladi.

CHIZIQLI TENGLAMA

$ax+b=0$ ko‘rinishdagi tenglamaga **chiziqli tenglama** deyiladi.

1. $a \neq 0,\ b \in R$ da, $x = -\frac{b}{a}$ - yagona yechim.

2. $a = 0,\ b \neq 0$ da, $x \in \varnothing$ – yechim yo‘q.

3. $a = 0,\ b = 0$ da, $x \in R$ – yechim cheksiz ko‘p bo‘ladi.

KVADRAT TENGLAMA

$ax^2+bx+c=0$, $a\neq 0$ tenglamaga **kvadrat tenglama** deb aytiladi.

1. Kvadrat tenglamadan to‘la kvadratga ajratish: $ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$

2. $D=b^2-4ac$ kvadrat tenglama **diskriminanti** deyiladi.

3. Tenglama ildizlari quyidagi formula bilan topiladi: $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}$.

4. Agar ax^2+bx+c kvadrat uchhadning diskriminanti manfiy bo‘lmasa, u holda, $ax^2+bx+c=a(x-x_1)(x-x_2)$ o‘rinlidir, bu yerda x_1,x_2– kvadrat uchhad nollari.

5. Kvadrat tenglama ildizlarining xossalari:

$D=b^2-4ac>0$ bo‘lsa, tenglama ikkita haqiqiy ildizga ega.

$D=b^2-4ac=0$ bo‘lsa, tenglama ikkita bir xil haqiqiy ildizga ega.

$D=b^2-4ac<0$ bo‘lsa, tenglama haqiqiy ildizga ega emas (ikkita har xil kompleks ildizlarga, ya’ni qo‘shma komleks ildizlarga ega).

VIYET TEOREMASI.

1. $x^2+px+q=0$ ko‘rinishdagi tenglamaga, **keltirilgan kvadrat tenglama** deyiladi. x_1, x_2 ildizlari bo‘lsa, quyidagi tengliklar bajariladi:

$$\begin{cases} x_1+x_2=-p \\ x_1x_2=q. \end{cases}$$

2. $ax^2+bx+c=0$ bo‘lsa, quyidagi tengliklar bajariladi:

1) $\begin{cases} x_1+x_2=-\frac{b}{a} \\ x_1\cdot x_2=\frac{c}{a}. \end{cases}$ 2) $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=\frac{b^2-2ac}{a^2}$. 3) $\frac{1}{x_1^2}+\frac{1}{x_2^2}=\frac{b^2-2ac}{c^2}$.

4) $x_1^3+x_2^3=(x_1+x_2)^3-3x_1x_2(x_1+x_2)=-\left(\frac{b}{a}\right)^3+\frac{3bc}{a^2}$. 5) $\frac{1}{x_1^3}+\frac{1}{x_2^3}=\frac{-b^3+3abc}{c^3}$.

3. $x^3+ax^2+bx+c=0$ keltirilgan kubik tenglama deyiladi. x_1,x_2,x_3 - tenglama ildizlari bo‘lsa, quyidagi tengliklar bajariladi: $\begin{cases} x_1+x_2+x_3=-a \\ x_1x_2x_3=-c. \end{cases}$

4. $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$ keltirilgan n-darajali tenglama deyiladi. $x_1,x_2,\ldots,x_n$ tenglama ildizlari bo‘lsa, quyidagi tengliklar bajariladi: $\begin{cases} x_1+x_2+\ldots+x_n=-a_1 \\ x_1x_2\cdot\ldots\cdot x_n=(-1)^n a_n. \end{cases}$

FUNKSIYA

A va B haqiqiy sonlar to‘plamlari berilgan bo‘lsin. A to‘plam elementlarini x, B to‘plam elementlarini y bilan belgilaylik.

A to‘plamning har bir elementiga B to‘plamning faqat bitta elementining mos qo‘yilish qonuniyati **funksiya** deyiladi va $y=f(x)$ deb belgilanadi. Bu yerda x **erkli o‘zgaruvchi (yoki argument)**, y **esa erksiz o‘zgaruvchi (yoki funksiya)** deb yuritiladi. x ning qabul qiladigan qiymatlari shu funksiyaning

aniqlanish sohasi, y ning qabul qiladigan qiymatlari esa funksiyaning o‘zgarish sohasi yoki **qiymatlar to‘plami** deyiladi.

Funksiya quyidagicha berilishi mumkin:

1) Jadval ko‘rinishida;
2) Nuqtalarning tartiblangan juftlari orqali;
3) Grafik tarzda;
4) Analitik usulda.

FUNKSIYALAR TO‘G‘RISIDA BA’ZI MA’LUMOTLAR

Agar $y=f(x)$ funksiyaning grafigi berilgan bo‘lsa, ($a>0$):

$y=f(x+a)$ – funksiyaning grafigi $y=f(x)$ funksiyaning grafigini x o‘qi bo‘ylab a birlik chapga parallel ko‘chirish bilan hosil qilinadi.

$y=f(x-a)$ – funksiyaning grafigi $y=f(x)$ funksiyaning grafigini x o‘qi bo‘ylab a birlik o‘ngga parallel ko‘chirish bilan hosil qilinadi.

$y=f(x)+a$ – funksiyaning grafigi $y=f(x)$ funksiyaning grafigini y o‘qi bo‘ylab a birlik yuqoriga parallel ko‘chirish bilan hosil qilinadi.

$y=f(x)-a$ – funksiyaning grafigi $y=f(x)$ funksiyaning grafigini y o‘qi bo‘ylab a birlik pastga parallel ko‘chirish bilan hosil qilinadi.

Ox o‘qiga nisbatan $y=f(x)$ ga simmetrik $y=-f(x)$ funksiya bo‘ladi.
Misol: $y=2x+3$ ga simmetrik $y=-2x-3$ bo‘ladi.

Oy o‘qiga nisbatan $y=f(x)$ ga simmetrik $y=f(-x)$ funksiya bo‘ladi.
Misol: $y=-3x+1$ ga simmetrik $y=3x+1$ bo‘ladi.

FUNKSIYANING ANIQLANISH SOHASI.

1. $y=\sqrt[m]{f(x)}$, m juft bo‘lsa, aniqlanish sohasi: $f(x)\geq 0$ tengsizlikni qanoatlantiruvchi nuqtalar to‘plami bo‘ladi.

2. $y=\frac{g(x)}{f(x)}$ bo‘lsa, aniqlanish sohasi: $f(x)\neq 0$ tengsizlikni qanoatlantiruvchi nuqtalar to‘plami va $f(x)$, $g(x)$ funksiyalar aniqlanish to‘plami kesishmasidan iborat bo‘ladi.

3. $\log_{g(x)} f(x)$ bo‘lsa, aniqlanish sohasi: $\begin{cases} f(x)>0 \\ g(x)>0 \\ g(x)\neq 1 \end{cases}$ tengsizliklar sistemasini qanoatlantiruvchi nuqtalar to‘plami bo‘ladi.

4. $y=\arcsin f(x)$ yoki $y=\arccos f(x)$ bo‘lsa, aniqlanish sohasi $|f(x)|\leq 1$ tengsizlikni qanoatlantiruvchi nuqtalar to‘plami bo‘ladi.

5. $y=tgx$ bo‘lsa, aniqlanish sohasi $x\neq\frac{\pi}{2}+n\pi,\quad n\in Z$, tengsizliklarni qanoatlantiruvchi nuqtalar to‘plami bo‘ladi.

6. $y=ctgx$ bo‘lsa, aniqlanish sohasi $x\neq n\pi,\quad n\in Z$, tengsizliklarni qanoatlantiruvchi nuqtalar to‘plami bo‘ladi.

FUNKSIYANING QIYMATLAR SOHASI

1. $y=\sqrt{ax^2+bx+c}$ funksiyada agar, *a>0, D≤0* bo‘lsa, qiymatlar sohasi: $E(y)=[\sqrt{y_0};+\infty)$; agar *a<0, D>0* bo‘lsa, $E(y)=[0;\sqrt{y_0}]$; agar *a>0, D≥0* bo‘lsa, qiymatlar sohasi: $E(y)=[0;+\infty)$; agar *a<0, D<0* bo‘lsa, $E(y)=\varnothing$ bo‘ladi, bu yerda $y_0=\dfrac{4ac-b^2}{4a}$.

2. $y=a\cos k(x)+b\cos k(x)$ bo‘lsa, qiymatlar sohasi: $E(y)=[-\sqrt{a^2+b^2};\sqrt{a^2+b^2}]$ bo‘ladi.

FUNKSIYANING JUFT-TOQLIGI.

1. Funksiyaning aniqlanish sohasi *x=0* ga nisbatan simmetrik bo‘lib, *y(-x)=y(x)* bo‘lsa, *y=y(x)* funksiya juft funksiya; *y(-x)=-y(x)* bo‘lsa, *y=y(x)* funksiya toq funksiya deyiladi; aks holda funksiya juft ham emas, toq ham emas deyiladi.
2. Juft=J; Toq=T belgilash kiritamiz: J$\pm$J=J.
3. J$\pm$T=Juft ham, toq ham emas (JTE).
4. JJ=J, J:J=J.
5. JT=T, J:T=T.
6. T$\pm$T=T.
7. TT=J, T:T=J.
8. Juft funksiyaning grafigi OY o‘qiga nisbatan simmetrik bo‘ladi.
9. Toq funksiyaning grafigi koordinata boshiga nisbatan simmetrik bo‘ladi.

Masalan: 1) *y=cosx* – juft funksiya, *y=sinx* - toq funksiya.
2) *y=arcsinx* va *y=arctgx* – toq funksiyalar.
3) *y=arccosx* va *y=arcctgx* –juft ham, toq ham emas.

FUNKSIYALARNING DAVRIYLIGI.

Shunday $T>0$ soni mavjud bo‘lsaki, *y=f(x)* funksiyaning aniqlanish sohasiga tegishli har bir *x* nuqta bilan bir qatorda *x-T* va *x+T* nuqtalar ham shu sohaga tegishli bo‘lib, *f(x+T)= f(x-T)=f(x)* tengliklar bajarilsa, u holda *f(x)* funksiya **davriy funksiya** deyiladi. Bunday xossali son, umuman olganda bir qancha bo‘lishi mumkin. Agar ularning ichida eng kichik musbat T_0 mavjud bo‘lsa, T_0 sonni *f(x)* funksiyaning **davri** deb aytamiz, *f(x)* funksiya esa, T_0 davrli funksiya deyiladi.

1. *y=cosx* va *y=sinx* funksiyalarning eng kichik musbat davri (e.k.m.) *T=2π.*
2. *y=tgx* va *y=ctgx* funksiyalarning e.k.m. davri *T= π.*
3. *y=sinkx* (*y=coskx*) ning e.k.m. davri $T=(2\pi/|k|),\ k\neq 0$.
4. *y=tgkx* (*y=ctgkx*) ning e.k.m. davri $T=(\pi/|k|),\ k\neq 0$.
5. Bir necha davriy funksiyalarning yig‘indisidan iborat davriy funksiyaning e.k.m. davrini topish uchun qo‘shiluvchi funksiyalarning e.k.m. davrlarini EKUK ni topish kerak.

Misol. $y=\sin\frac{x}{2}+ctg3x=y_1+y_2$, $y_1=\sin\frac{x}{2}$ ning davri $T_1=\dfrac{2\pi}{\frac{1}{2}}=4\pi$, $T_1=4\pi$, $y=ctg3x$ ning davri $T_2=\frac{\pi}{3}$, $EKUK(T_1;T_2)=EKUK\left(4\pi;\frac{\pi}{3}\right)=\frac{\pi}{3}\cdot EKUK(12;1)=\frac{\pi}{3}\cdot 12=4\pi$.

FUNKSIYAGA TESKARI BO‘LGAN FUNKSIYANI TOPISH.

$y = f(x)$ funksiyaga teskari funksiyani topish uchun:

1) $y = f(x)$ tenglikdan x topiladi;
2) hosil bo‘lgan tenglikda x va y ning o‘rni o‘zaro almashtiriladi;
3) funksiyaning aniqlanish sohasi hisobga olinadi.

Misol: $y = \frac{2}{x-1} + 3$ funksiyaga teskari funksiyani toping?

1) $\frac{2}{x-1} = y - 3 \Rightarrow x - 1 = \frac{2}{y-3} \Rightarrow x = \frac{2}{y-3} + 1$

2) $x \leftrightarrow y \Rightarrow y = \frac{2}{x-3} + 1$ 3) $y \neq 1$

$y = f(x)$ funksiya va unga teskari funksiya grafigi *y*=*x* to‘g‘ri chiziqqa nisbatan simmetrik bo‘ladi.

FUNKSIYANING O‘SISHI VA KAMAYISHI.

$f(x)$ funksiya *(a,b)* oraliqda aniqlangan bo‘lib, ixtiyoriy $x_1, x_2 \in (a,b)$ $x_1 > x_2$ lar uchun, agar $y(x_1) > y(x_2)$ bo‘lsa, $y = f(x)$ funksiya *(a,b)* oraliqda o‘suvchi; agar $y(x_1) < y(x_2)$ bo‘lsa, $y = f(x)$ funksiya *(a,b)* oraliqda kamayuvchi funksiya deyiladi.

CHIZIQLI FUNKSIYA

y=*kx*+*b* $(k, b \in R)$ ko‘rinishdagi funksiyaga **chiziqli funksiya** deyiladi.

Bu funksiya grafigi *Oxy* tekisligida to‘g‘ri chiziqni ifodalaydi, shuning uchun u chiziqli funksiya deyiladi.

CHIZIQLI FUNKSIYANING Xossalari

1. Funksiyaning aniqlanish sohasi $D(y) = R$ dan iborat.
2. Agar $k \neq 0$ bo‘lsa, chiziqli funksiyaning qiymatlar (o‘zgarish) sohasi barcha haqiqiy sonlardan iborat: $E(y) = R$. $k = 0$ bo‘lsa, $E(y) = \{b\}$.
3. $k \neq 0$ bo‘lsa, $x_0 = -\frac{b}{k}$ funksiya noli bo‘ladi. Agar $k > 0$ bo‘lsa, bu funksiya $(x_0; +\infty)$ da musbat ishorali, sohaning $(-\infty; x_0)$ qismida manfiy ishoralidir. $k < 0$ bo‘lsa, aksincha $(x_0; +\infty)$ sohada manfiy, sohaning qolgan qismida musbat; $k = 0$ bo‘lgan holda bu funksiya $(-\infty, +\infty)$ sohada b ning ishorasi kabi ishorali bo‘ladi.
4. $x \in (-\infty; +\infty)$ oraliqda *k*>0 bo‘lsa, o‘suvchi, *k*<0 bo‘lsa, kamayuvchi.
5. $b \neq 0$, $k \neq 0$ bo‘lganda bu funksiya toq ham emas, juft ham emas; $b = 0$, $k \neq 0$ bo‘lgan holda esa toq funksiyadir va $y = kx$ funksiyadagi *x* va *y* miqdorlarga o‘zaro to‘g‘ri proporsional ravishda bog‘langan miqdorlar, k –proporsionallik koeffisiyenti deyiladi.
6. *b=0, k>0, y=kx* – to‘g‘ri chiziq koordinata boshidan o‘tadi. (I, III choraklar).
7. *b=0, k<0, y=kx* – to‘g‘ri chiziq koordinata boshidan o‘tadi. (II, IV choraklar).
8. *b>0, k=0, y=b* – to‘g‘ri chiziq absissalar o‘qiga parallel. (I, II choraklar).
9. *b<0, k=0, y=b* – to‘g‘ri chiziq absissalar o‘qiga parallel. (III, IV choraklar).

10. $kx+b=0$, $x=-b/k$, $b<0$, $k>0$ ($b>0$, $k<0$) – to‘g‘ri chiziq ordinata o‘qiga parallel (I, IV choraklar).
11. $kx+b=0$, $x=-b/k$, $b<0$, $k<0$ ($b>0$, $k>0$) – to‘g‘ri chiziq ordinata o‘qiga parallel (II, III choraklar).
12. $k>0$, $b>0$, $y=kx+b$ – to‘g‘ri chiziq I, II, III choraklardan o‘tadi.
13. $k>0$, $b<0$, $y=kx+b$ – to‘g‘ri chiziq I, III, IV choraklardan o‘tadi.
14. $k<0$, $b>0$, $y=kx+b$ – to‘g‘ri chiziq I, II, IV choraklardan o‘tadi.
15. $k<0$, $b<0$, $y=kx+b$ – to‘g‘ri chiziq II, III, IV choraklardan o‘tadi.
16. $y=kx+b$ to‘g‘ri chiziqqa perpendikulyar va $M(x_0;y_0)$ nuqtadan o‘tgan to‘g‘ri chiziq tenglamasi: $y-y_0=-\frac{1}{k}(x-x_0)$ ($k_1=-1/k$ – perpendikulyarlik sharti).
17. $y=kx+b$ to‘g‘ri chiziqqa parallel va $M(x_0;y_0)$ nuqtadan o‘tgan to‘g‘ri chiziq tenglamasi $y-y_0=k(x-x_0)$ ($k_1=k$ – parallellik sharti).
18. $y=k_1x+b_1$ va $y=k_2x+b_2$ to‘g‘ri chiziqlar orasidagi burchak quyidagi formulalar orqali topiladi:

$$tg\alpha=k_1,\quad tg\beta=k_2,\quad \varphi=|\alpha-\beta|,\quad tg\varphi=\frac{k_2-k_1}{1+k_1k_2}\ yoki\ \ tg\varphi=\frac{k_1-k_2}{1+k_1k_2}$$

19. $A_1x+B_1y+C_1=0$ va $A_2x+B_2y+C_2=0$ bo‘lsa, to‘g‘ri chiziqlar orasidagi burchak quyidagi formula yordamida topiladi:

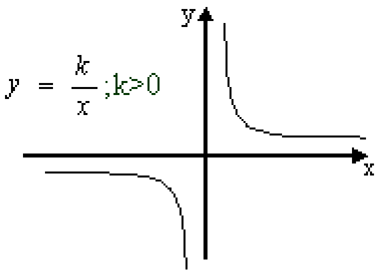
$$tg\alpha=\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}\ \text{yoki}\ tg\alpha=\frac{A_2B_1-A_1B_2}{A_1A_2+B_1B_2}.$$

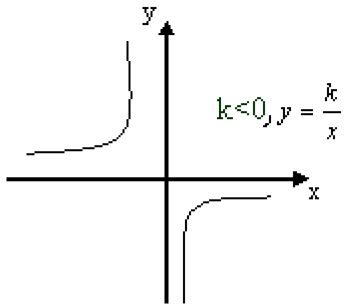
20. $\frac{A_1}{A_2}=\frac{B_1}{B_2}$ parallellik sharti.
21. $A_1A_2+B_1B_2=0$ perpendikulyarlik sharti.

TESKARI PROPORSIONALLIK.

$y=\frac{k}{x}$ funksiyaga teskari proporsionallik deyiladi. ($k\in R, k\neq 0$)

1. Funksiyaning aniqlanish sohasi $x\neq 0$ yoki $D(y)=(-\infty,0)\cup(0;\infty)$ bo‘ladi.
2. Qiymatlar sohasi esa $E(y)\in(-\infty,0)\cup(0;\infty)$ dan iborat.
3. $y=\frac{k}{x}$ toq funksiya
4. k>0 bo‘lsa, $y=\frac{k}{x}$ funksiya $(-\infty;0)$ va $(0;+\infty)$ oraliqlarning har birida kamayuvchi, k<0 bo‘lsa, $y=\frac{k}{x}$ funksiya $(-\infty;0)$ va $(0;+\infty)$ oraliqlarning har birida o‘suvchi bo‘ladi.
5. Grafigi





6. Teskari proportsionallik grafigi giperbola deyiladi.

7. $y=\frac{k}{x}$ funksiyadagi x va y miqdorlarga o'zaro teskari proporsional ravishda bog'langan miqdorlar, k – teskari proporsionallik koeffisiyenti deyiladi.

CHIZIQLI TENGSIZLIKLAR

1. $ax+b>0 \Leftrightarrow \begin{cases} x>-\frac{b}{a}, \text{ agar } a>0 \text{ bo'lsa, } x\in\left(-\frac{b}{a};+\infty\right), \\ x<-\frac{b}{a}, \text{ agar } a<0 \text{ bo'lsa, } x\in\left(-\infty;-\frac{b}{a}\right). \end{cases}$

2. $ax+b<0 \Leftrightarrow \begin{cases} x<-\frac{b}{a}, \text{ agar } a>0 \text{ bo'lsa, } x\in\left(-\infty;-\frac{b}{a}\right), \\ x>-\frac{b}{a}, \text{ agar } a<0 \text{ bo'lsa, } x\in\left(-\frac{b}{a};+\infty\right). \end{cases}$

TENGSIZLIK XOSSALARI

1. Agar $f(x)\geq g(x)$ bo'lsa, $c>0$ da $cf(x)\geq cg(x)$; $c<0$ da $cf(x)\leq cg(x)$ bo'ladi;
2. Agar $|f(x)|\geq|g(x)|$ $(|f(x)|\leq|g(x)|)$ bo'lsa, u holda $f^{2n}(x)\geq g^{2n}(x)$ $(f^{2n}(x)\leq g^{2n}(x))$ bo'ladi;
3. Agar $|f(x)|\leq c$, $c>0$ bo'lsa, u holda $-c\leq f(x)\leq c$ bo'ladi.

CHIZIQLI TENGLAMALAR SISTEMASI.

$\begin{cases} a_1x+b_1y=c_1, \\ a_2x+b_2y=c_2, \end{cases}$ yoki $\begin{cases} y=\frac{c_1}{b_1}-\frac{a_1}{b_1}x \\ y=\frac{c_2}{b_2}-\frac{a_2}{b_2}x \end{cases}$

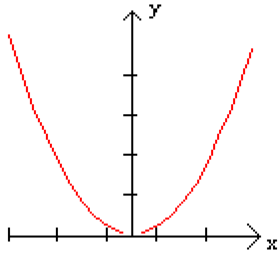
1) $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ to'g'ri chiziqlar parallel bo'ladi, tenglamalar sistemasi yechimga ega emas. $x\in\varnothing$, $y\in\varnothing$.

2) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ to'g'ri chiziqlar ustma –ust tushadi va tenglamalar sistemasi cheksiz ko'p yechimga ega.

3) $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ to'g'ri chiziqlar kesishadi va tenglamalar sistemasi yagona yechimga ega.

KVADRAT FUNKSIYA

$f(x) = ax^2 + bx + c$ $(a, b, c \in R,\ a \neq 0)$ ko'rinishdagi funksiyaga kvadratik funksiya deyiladi. Uni $f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$, $D = b^2 - 4ac$ shaklda ifodalash mumkin. Ifodadagi D uning diskriminanti. Bu funksiya grafigi parabola deb ataluvchi egri chiziqdan iborat.



$y=x^2$ funksiya

1. Aniqlanish sohasi: $D(y) = (-\infty, \infty)$.
2. Qiymatlar sohasi: $E(y) = [0, \infty)$.
3. $y = x^2$ juft funksiya.
4. funksiya $(-\infty, 0]$ oraliqda kamayadi, $[0, \infty)$ oraliqda o'sadi.
5. $y = x^2$ funksiyaning grafigi parabola.

$y=ax^2+bx+c$ funksiya

Funksiyaning umumiy ko'rinishi: $y=ax^2+bx+c=a(x-x_0)^2+y_0$; a,b,c – koeffisiyentlar, $x_0 = -\frac{b}{2a}$; $y_0 = \frac{4ac - b^2}{4a}$.

1. Aniqlanish sohasi: $x \in (-\infty, \infty)$, $D(y) = (-\infty, \infty)$.
2. Qiymatlar sohasi: $a>0$ da $E(y)=[y_0, \infty)$, $a<0$ da $E(y) = (-\infty, y_0]$.
3. Funksiya grafigi parabola, parabolaning uchi $\left(\frac{-b}{2a}; \frac{4ac - b^2}{4a}\right)$ nuqtada yotadi;
4. Simmetriya o'qi $x = \frac{-b}{2a}$.
5. $a<0$ bo'lsa, funksiya $(-\infty; -\frac{b}{2a}]$ oraliqda o'suvchi, $[-\frac{b}{2a}; \infty)$ oraliqda kamayuvchi.
6. $a>0$ bo'lsa, funksiya $(-\infty; -\frac{b}{2a}]$ oraliqda kamayuvchi, $[-\frac{b}{2a}; \infty)$ oraliqda o'suvchi.
7. $a>0$ bo'lsa, $y_{min} = \frac{4ac - b^2}{4a}$ eng kichik qiymatga ega; $a<0$ bo'lsa $y_{max} = \frac{4ac - b^2}{4a}$ eng katta qiymatga ega.
8. $ax^2+bx+c>0$, x_1; x_2 - $ax^2+bx+c=0$ tenglama ildizlari va $x_1 < x_2$.

a) $a>0$, $D>0$ bo'lsa, $x \in (-\infty, x_1) \cup (x_2; \infty)$

b) $a>0$, $D<0$ bo'lsa, $x \in (-\infty, \infty)$

c) $a<0$, $D>0$ bo'lsa, $x \in (x_1, x_2)$

d) $a<0$, $D<0$ bo'lsa, $x \in \varnothing$

9. $ax^2+bx+c<0$, x_1 ;x_2 - $ax^2+bx+c=0$ tenglama ildizlari.

a) $a>0, D>0$ bo‘lsa, $x \in (x_1, x_2)$

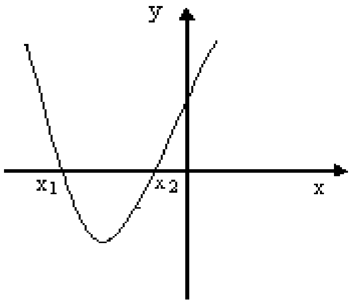
b) $a>0, D<0$ bo‘lsa $x \in \varnothing$

c) $a<0, D>0$ bo‘lsa $x \in (-\infty, x_1) \cup (x_2; \infty)$

d) $a<0, D<0$ bo‘lsa $x \in (-\infty, \infty)$

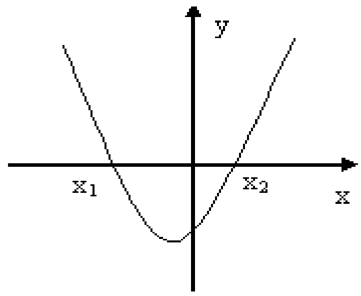
Grafiklari

1) $a>0, D>0, x_1<0, x_2<0.$



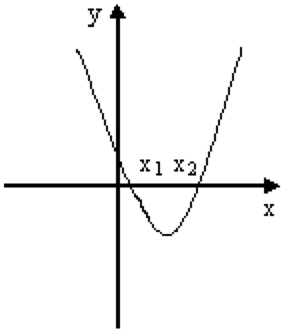
Grafigi I,II,III choraklarda.

2) $a>0, D>0, x_1<0, x_2>0.$



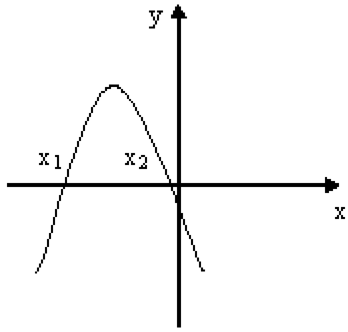
Grafigi I,II,III,IV choraklarda.

3) $a>0, D>0, x_1>0, x_2>0.$



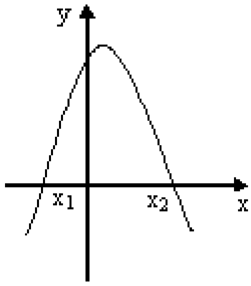
Grafigi I,II,IV choraklarda.

4) $a<0$ $D>0$ $x_1<0$ $x_2<0.$



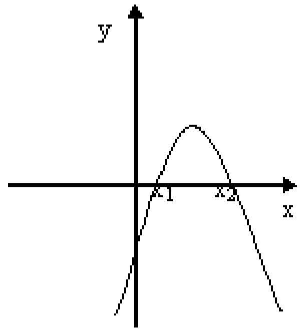
Grafigi II,III,IV choraklarda.

5) $a<0, D>0, x_1<0, x_2>0.$



Grafigi I,II,III,IV choraklarda.

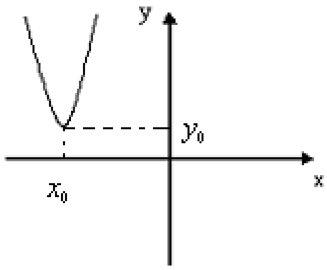
6) $a<0, D>0, x_1>0, x_2>0.$



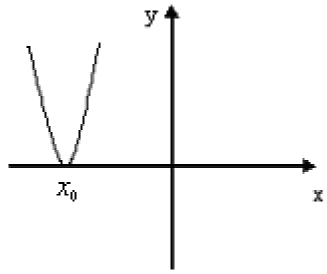
Grafigi I,III,IV choraklarda.

7) $a>0$, $D<0$, $x_0 = -\frac{b}{2a}$, $y_0 = \frac{4ac-b^2}{4a} > 0$.

8) $a>0, D=0, y_0=0.$

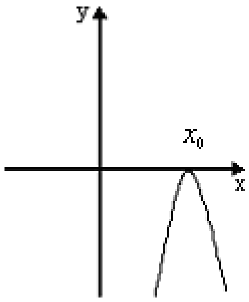


Grafigi I,II choraklarda.



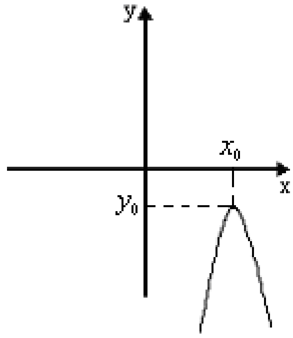
Grafigi I,II choraklarda.

9) *a*<0, *D*=0, y_0=0.



Grafigi III,IV choraklarda.

10) $a<0$ $D<0$, $x_0=-\frac{b}{2a}$, $y_0=\frac{4ac-b^2}{4a}<0$.

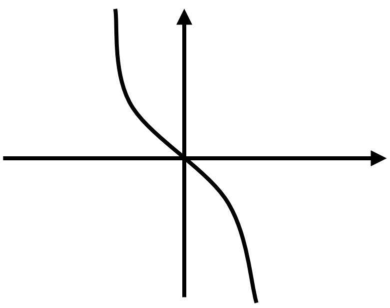


Grafigi III,IV choraklarda.

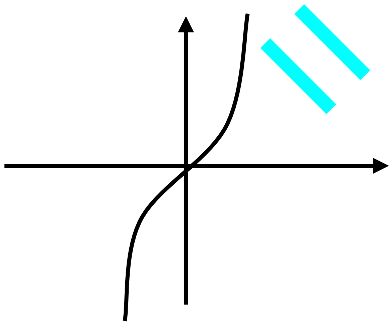
$Y=AX^3$ FUNKSIYA

1. Aniqlanish sohasi: $x \in (-\infty;+\infty)$.
2. Qiymatlar sohasi: $y \in (-\infty;+\infty)$.
3. a>0 da o'suvchi, a<0 da kamayuvchi.
4. $y=ax^3$ toq funksiya.
5. $y=ax^3$ funksiyaning grafigi kubik parabola.

1) $a<0$

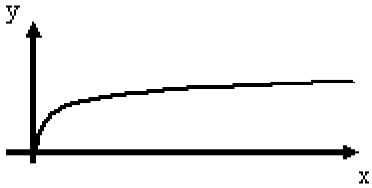


2) $a>0$



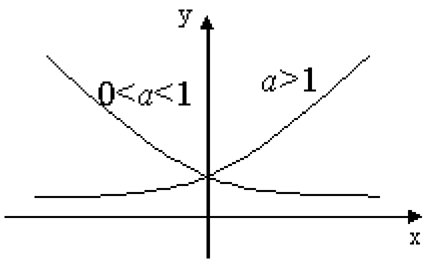
$y = ax^n$, $n \in R$ - funksiya **<u>darajali funksiya</u>** deyiladi.

$y=\sqrt{x}$ FUNKSIYA



1. Aniqlanish sohasi: $x \in [0;+\infty)$.
2. Qiymatlar sohasi: $y \in [0;+\infty)$.
3. $y = \sqrt{x}$ funksiya $x \in [0;+\infty)$ o‘suvchi.
4. $y = \sqrt{x}$ funksiyaning grafigi I chorakdagi yarim parabola.

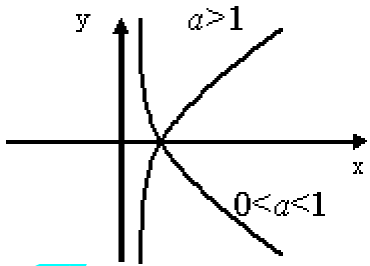
KO‘RSATKICHLI FUNKSIYA



1. Umumiy ko‘rinishi: $\boldsymbol{y=a^x}$ ($a>0$, $a\neq1$).
2. Aniqlanish sohasi: $x \in (-\infty;+\infty)$.
3. Qiymatlar sohasi: $y \in (0;+\infty)$.
4. $a>1$ bo‘lganda $\boldsymbol{y=a^x}$ funksiya aniqlanish sohasida
 o‘suvchi, $0<a<1$ bo‘lganda esa kamayuvchi.
5. Har doim (0;1) nuqtadan o‘tadi.

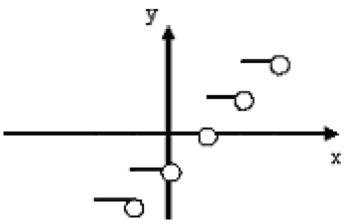
LOGARIFMIK FUNKSIYA

$y = a^x$ ko‘rsatkichli funksiyaga teskari bo‘lgan funksiya a asosli logarifmik funksiya deyiladi



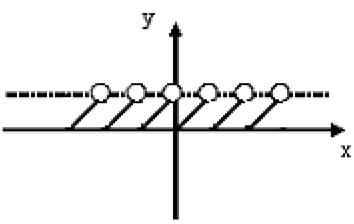
va $y=\log_a x$ ko‘rinishda yoziladi. Bunda $a>0$, $a\neq1$, $x>0$.
1. Aniqlanish sohasi: $x \in (0;+\infty)$
2. Qiymatlar sohasi: $y \in (-\infty;+\infty)$
3. $0<a<1$ bo‘lganda kamayuvchi, $a>1$ bo‘lganda o‘suvchi.
4. Har doim (1;0) nuqtadan o`tadi.

Y=[*X*] – SONNING BUTUN QISMI.



[*x*] –*x* dan oshmaydigan eng katta butun son.
Misollar: [2,31]=2; [14,3]=14; [-1,34]=-2; [-3,13]=-4.

Y={X} SONNING KASR QISMI.



$\{x\}=x-[x]$.
Misollar: {5,76}=0,76; {2,84}=0,84; {-5,26}=0,74; {-1,24}=0,76; {1,24}=0,24.

ARIFMETIK PROGRESSIYA.

Ikkinchi hadidan boshlab har bir hadi o‘zidan oldingi hadga bir xil sonni qo‘shish yordamida hosil qilingan sonlar ketma-ketligi arifmetik progressiya deyiladi. Demak, arifmetik progressiyaning har bir hadi o'zidan avvalgi haddan ayirma deb ataluvchi d songa farq qiladi. Arifmetik pogressiya hadlari $a_1, a_2, a_3, \dots, a_n$ kabi belgilanadi.

1. Ayirmani topish: $d = a_2 - a_1 = a_{n+1} - a_n = \dfrac{a_n - a_k}{n-k} \qquad (n > k, \qquad n, k \in N)$.

2. n -hadini topish formulasi: $a_n = a_1 + (n-1)d$.

3. O'rta hadini topish: $a_n = \dfrac{a_{n-1} + a_{n+1}}{2}, \ n > 1; \ a_n = \dfrac{a_{n-k} + a_{n+k}}{2}, \ n > k$.

4. Agar $a_n + a_m = a_k + a_p$ bo'lsa, $n+m=k+p$ tenglik o'rinli.

5. Arifmetik progressiyaning dastlabki n ta hadi yig'indisi topish formulasi:

$$S_n = \frac{a_1 + a_n}{2} n = \frac{a_2 + a_{n-1}}{2} n = \frac{2a_1 + (n-1)d}{2} n.$$

6. $a_n = S_n - S_{n-1}$.

7. $a, a+d, a+2d, \dots, a(n-1)d \qquad bo'lsa,$

$$S_n = a + 2(a+d) + \dots + n(a+(n-1)d) = \frac{n(n+1)}{6}(3a + 2(n-1)d)$$

8. $S_{n+k} = S_n + S_k + nkd.$

9. $S_{n+m} = \dfrac{m+n}{m-n}(S_m - S_n), \qquad m \neq n.$

10. $a, a+d, a+2d, \dots, a+(n-1)d \qquad va \qquad b, b+p, b+2p, \dots, b+(n-1)p$ bo'lsa,

$$S_n = ab + (a+d)(b+p) + (a+2d)(b+2p) + \dots + (a+(n-1)d)(b+(n-1)p) =$$
$$= a\left(\frac{2b-(n-1)p}{2}\right)n + bd\frac{n(n+1)}{2} + pd\frac{(n-1)n(2n-1)}{6}.$$

12. $S_n = a + \overline{aa} + \overline{aaa} + \dots + \overline{aa..a} = \dfrac{a}{9}\left(\dfrac{10^{n+1} - 10}{9} - n\right)$, bunda $n \in N, a \neq 0$ raqam.

GEOMETRIK PROGRESSIYA.

Birinchi hadi noldan farqli, ikkinchi hadidan boshlab har bir hadi avvalgi hadini nolga teng bo'lmagan bir xil songa ko'paytirlganiga teng bo'lgan sonlar ketma-ketligi geometrik progressiya deyiladi. Demak, geometrik progressiya shunday sonlar ketma-ketligi ekanki, uning har bir hadini o'zidan bir nomer avvalgi hadiga nisbati maxraj deb ataluvchi q - o'zgarmas songa teng bo'ladi. Geometrik progressiya hadlari $b_1, b_2, \dots, b_n$ kabi belgilanadi.

1. Maxrajini topish formulasi: $q = \dfrac{b_2}{b_1} = \dfrac{b_{n+1}}{b_n}, \qquad (q \neq 1).$

2. n- hadini topish formulasi: $b_n = b_1 \cdot q^{n-1}$.

3. O'rta hadini topish formulasi: $b_2 = \sqrt{b_1 \cdot b_3}, \qquad b_n = \sqrt{b_{n-1} \cdot b_{n+1}}, \ b_n > 0, \ n = 2,3,\dots.$

4. $b_m \cdot b_n = b_k \cdot b_p$ bo'lsa, $m+n=k+p$ o'rinli.

5. Dastlabki n ta hadining yig'indisi: $S_n = \dfrac{b_n q - b_1}{q-1} = \dfrac{b_1(q^n - 1)}{q-1}; \qquad (q \neq 1).$

6. $b_n = S_n - S_{n-1}$.

CHEKSIZ KAMAYUVCHI GEOMETRIK PROGRESSIYA.

$b_1, b_1q, b_1q^2, ..., b_1q^{n-1}, ...$ geometrik progressiya berilgan bo‘lsin.

Agar $|q|<1$ bo‘lsa, bu ketma-ketlik cheksiz kamayuvchi geometrik progressiya deyiladi.

$S = b_1 + b_1q + b_1q^2 + ... + b_1q^{n-1} + ...$ yig‘indisini topish formulasi:

$$S = \frac{b_1}{1-q}, \quad (|q|<1).$$

LOGARIFMLAR

$y = \log_a x \iff a^y = x$, $(a>0,\ a\neq 1)$ bo‘lib, bundan $a^{\log_a x} = x$ logarifmning asosiy ayniyati kelib chiqadi.

$\log_{10} b = \lg b -$ o'nli logarifm; $\log_e b = \ln b -$ natural logarifm - umumiy belgilashlar qabul qilingan.

Logarifmning xossalari:

Hamma holda $a>0,\ a\neq 1,\ x>0,\ y>0,\ c>0,\ c\neq 1.$

$1. \log_a 1 = 0;$ $\quad 2. \log_a a = 1;$ $\quad 3. \log_a xy = \log_a x + \log_a y;$

$4. \log_a \frac{x}{y} = \log_a x - \log_a y;$ $\quad 5. \log_a b^n = n\cdot \log_a b;$

$6. \log_a b = \frac{\log_c b}{\log_c a};$

$7. \log_{a^k} b^k = \log_a b;$ $\quad 8. \log_{a^k} b^n = \frac{n}{k}\log_a b;$ $\quad 9. \log_a \sqrt[m]{b^n} = \frac{n}{m}\log_a b;$

$10.\ a^{\log_c b} = b^{\log_c a};$ $\quad 11.\ \log_a b = \frac{1}{\log_b a};$ $\quad 12.\ \log_a b^{\log_a b} = (\log_a b)^2 = \log_a^2 b$

$13.\ \log_a b \cdot \log_b c \cdot ... \cdot \log_n m = \log_a m;$ $14.\ \log_a x \cdot \log_b y = \log_a y \cdot \log_b x;$

$15.\ a>1;\ 0<b<1$ yoki $0<a<1,\ b>1$ da $\log_a b<0;$

$16.\ a>1,\ b>1$ yoki $0<a<1,\ 0<b<1$ da $\log_a b>0;$

$17.\ a>1,\ b>c>0$ da $\log_a b > \log_a c;$

$18.\ 0<a<1,\ b>c>0$ da $\log_a b < \log_a c;$

$19.\ b>a>1$ bo'lsin, agar $p>1,$ bo'lsa, $\log_a p > \log_b p$ bo'ladi; agar $0<p<1$ bo'lsa $\log_a p < \log_b p$ bo'ladi;

$20.\ 0<a<b<1$ bo'lsin, agar $0<p<1$ bo'lsa, $\log_a p < \log_b p$ bo'ladi. Agar $p>1$ bo'lsa, $\log_a p > \log_b p$ bo'ladi.

$21.\ a>b>0$ bo'lsin, agar $p>1$ bo'lsa, $\log_p a > \log_p b$ bo'ladi. Agar $0<p<1$ bo'lsa, $\log_p a > \log_p b$ bo'ladi.

$22.\ a^{f(x)} > a^{g(x)}$ tengsizlik, $a>1$ da $\begin{cases} f(x) > g(x), \\ f(x)>0, \\ g(x)>0 \end{cases}$ tengsizliklar sistemasiga; $0<a<1 \ da \ \begin{cases} f(x)<g(x), \\ f(x)>0, \\ g(x)>0 \end{cases}$ tengsizliklar sistemasiga teng kuchli bo‘ladi.

23. $\log_a f(x)<c$ tengsizlik, $a>1 \ da \ \begin{cases} f(x)>0 \\ f(x)<a^c \end{cases}$ tengsizliklar sistemasiga;

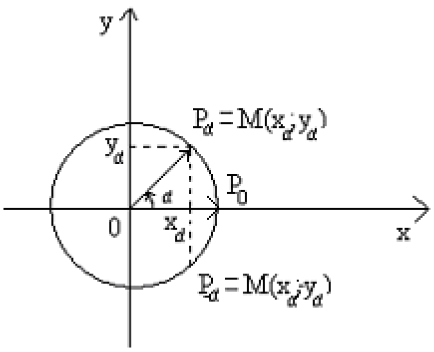
0<a<1 da $\begin{cases} f(x)>0 \\ f(x)>a^c \end{cases}$ tengsizliklar sistemasiga teng kuchli bo‘ladi.

24. Har qanday natural n son uchun $\log_n(n+1)>\log_{(n+1)}(n+2)$ o‘rinli.

IXTIYORIY BURCHAK TRIGONOMETRIK FUNKSIYALARI

$\alpha^0=\frac{180^0}{\pi}\cdot a_{rad}$ radian o‘lchov birligidan gradusga o‘tish

$a_{rad}=\frac{\pi}{180^0}\cdot\alpha^0$ gradus o‘lchov birligidan radianga o‘tish.



Markazi koordinatalar boshida bo‘lgan, radiusi 1 ga teng aylana berilgan bo‘lsin. Koordinata boshi atrofida OP_0 vektorni soat strelkasi harakatiga teskari yo‘nalishda biror burchakka burish $P_0=M(1;0)$ nuqtani $P_\alpha=M(x_\alpha;y_\alpha)$ nuqtaga o‘tkazsin. Bu musbat yo‘nalishli burilish hisoblanadi. Soat strelkasi harakati bo‘ylab α burchakka burish manfiy burchakka burish hisoblanadi. Bu holda α o‘zgarishi bilan P_α nuqtaning koordinatalari x_α va y_α lar ham turlicha o‘zgaradi.

$OP_0=(1;0)$ vektorni α burchakka burish bilan hosil qilingan $OP_\alpha=(x_\alpha;y_\alpha)$ vektorning absissasi α burchakning kosinusi, ordinatasi esa uning sinusi deb aytiladi va mos ravishda $x_\alpha=\cos\alpha$, $y_\alpha=\sin\alpha$ deb belgilanadi.

ASOSIY TRIGONOMETRIK AYNIYATLAR

1. $\sin^2\alpha+\cos^2\alpha=1;$ 2. $tg\alpha=\frac{\sin\alpha}{\cos\alpha};$ 3. $ctg\alpha=\frac{\cos\alpha}{\sin\alpha};$

4. $tg\alpha\cdot ctg\alpha=1;$ 5. $1+tg^2\alpha=\frac{1}{\cos^2\alpha};$ 6. $1+ctg^2\alpha=\frac{1}{\sin^2\alpha};$

TRIGONOMETRIK FUNKSIYALARNI BIRINI IKKINCHISI ORQALI IFODALASH

1. $\sin\alpha=\pm\sqrt{1-\cos^2\alpha}=\frac{tg\alpha}{\pm\sqrt{1+tg^2\alpha}}=\frac{1}{\pm\sqrt{1+ctg^2\alpha}};$

2. $\cos\alpha=\pm\sqrt{1-\sin^2\alpha}=\frac{1}{\pm\sqrt{1+tg^2\alpha}}=\frac{ctg\alpha}{\pm\sqrt{1+ctg^2\alpha}};$

3. $tg\alpha = \dfrac{\sin\alpha}{\pm\sqrt{1-\sin^2\alpha}} = \dfrac{\pm\sqrt{1-\cos^2\alpha}}{\cos\alpha} = \dfrac{1}{ctg\alpha}$;

4. $ctg\alpha = \dfrac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}} = \dfrac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha} = \dfrac{1}{tg\alpha}$.

Bu yerda α ning qaysi chorakka tegishli ekanligiga qarab “-”, “+” ishoralidan biri olinadi. Masalan: Agar $\alpha \in \left(0, \dfrac{\pi}{2}\right)$, ya‘ni birinchi chorakka tegishli qiymat qabul qilsa, 1-formulada “+”ishorasi, $\alpha \in \left(\pi, \dfrac{3\pi}{2}\right)$, ya‘ni III chorakka tegishli qiymat qabul qilsa, 1-formulada “-” ishorasi olinadi.

BURCHAKLAR YIG‘INDISI VA AYIRMASINING TRIGONAMETRIK FUNKSIYALARI

1. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$.
2. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.
3. $tg(\alpha \pm \beta) = \dfrac{tg\alpha \pm tg\beta}{1 \mp tg\alpha \cdot tg\beta}$.

3'. $ctg(\alpha \pm \beta) = \dfrac{ctg\alpha \cdot ctg\beta \mp 1}{ctg\alpha \pm ctg\beta}$.

4. $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$.
5. $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin\beta\cos\alpha$.
6. $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$.
7. $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$.
8. $\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma$.
9. $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$.

IKKILANGAN ARGUMENTNING TRIGONOMETRIK FORMULALARI

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha = \dfrac{2tg\alpha}{1+tg^2\alpha}$. 2. $tg2\alpha = \dfrac{2tg\alpha}{1-tg^2 a}$.

3. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \dfrac{1-tg^2 a}{1+tg^2 a}$.

YARIM BURCHAKNING TRIGONOMETRIK FUNKSIYALARI

1. $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$. **2.** $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$.

3. $tg\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$.

4. $ctg\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}} = \dfrac{1+\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1-\cos\alpha}$.

5. $\sin\alpha = \dfrac{2tg\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}}$ $=1-2\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

6. $\cos\alpha = \dfrac{1-tg^2\frac{\alpha}{2}}{1+tg^2\frac{\alpha}{2}}$ $=1-2\sin^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

7. $\mathrm{tg}\alpha = \dfrac{2tg\frac{\alpha}{2}}{1-tg^2\frac{\alpha}{2}}$. **8.** $\mathrm{ctg}\alpha = \dfrac{ctg^2\frac{\alpha}{2}-1}{2ctg\frac{\alpha}{2}} = \dfrac{1-tg^2\frac{\alpha}{2}}{2tg\frac{\alpha}{2}}$.

TRIGONOMETRIK FUNKSIYALAR YIG'INDISI VA AYIRMASI UCHUN FORMULALAR

1. $\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cdot\cos\frac{\alpha\mp\beta}{2}$.

2. $\sin\alpha + \cos\alpha = \sqrt{2}\cos(45^0-\alpha) = \sqrt{2}\sin(45^0+\alpha)$.

3. $\cos\alpha - \sin\alpha = \sqrt{2}\sin(45^0-\alpha) = \sqrt{2}\cos(45^0+\alpha)$.

4. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}$.

5. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$. 6. $tg\alpha \pm tg\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$.

7. $\mathrm{ctg}\alpha + \mathrm{ctg}\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\cdot\sin\beta}$. 8. $1+\sin\alpha = 2\cos^2\left(45^0-\frac{\alpha}{2}\right)$.

9. $1-\sin\alpha = 2\sin^2\left(45^\circ-\frac{\alpha}{2}\right)$. 10. $1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$.

11. $\mathrm{tg}\alpha + \mathrm{ctg}\beta = \frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}$. 12. $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$.

13. $1-\mathrm{tg}^2\alpha = \frac{\cos2\alpha}{\cos^2\alpha}$. 14. $1-\mathrm{ctg}^2\alpha = -\frac{\cos2\alpha}{\sin^2\alpha}$.

15. $p\cos\alpha + q\sin\alpha = r\sin(\beta+\alpha)$, bu yerda: $r=\sqrt{p^2+q^2}$, $\sin\beta = \frac{p}{r}$, $\cos\beta = \frac{q}{r}$.

TRIGONOMETRIK FUNKSIYLAR UCHUN DARAJANI PASAYTIRISH FORMULALARI

1. $\cos^2\alpha = \frac{1+\cos2\alpha}{2}$. **2.** $\sin^2\alpha = \frac{1-\cos2\alpha}{2}$. **3.** $\cos^3\alpha = \frac{3\cos\alpha+\cos3\alpha}{4}$.

4. $\sin^3\alpha = \frac{3\sin\alpha-\sin3\alpha}{4}$. **5.** $\sin^4\alpha - \cos^4\alpha = -\cos2\alpha$.

6. $\sin^4\alpha = \frac{1}{8}(\cos4\alpha - 4\cos2\alpha + 3)$.

7. $\cos^4\alpha = \frac{1}{8}(\cos4\alpha + 4\cos2\alpha + 3)$.

8. $\sin^4\alpha + \cos^4\alpha = 1 - 2\sin^2\alpha \cdot \cos^2\alpha = \frac{3}{4} + \frac{1}{4}\cos 4\alpha$.

9. $\sin^6\alpha + \cos^6\alpha = 1 - 3\sin^2\alpha \cos^2\alpha = \frac{5}{8} + \frac{3}{8}\cos 4\alpha = 1 - \frac{3}{4}\sin^2 2\alpha = \frac{1 + 2\cos^2 2\alpha}{4}$.

10. $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$.

11. $\cos^8\alpha - \sin^8\alpha = \frac{1}{4}(\cos 2\alpha\ (3 + \cos 4\alpha))$.

TRIGONOMETRIK FUNKSIYALARNING KO‘PAYTMASINI YIG‘INDIGA KELTIRISH

1. $\sin\alpha \cdot \cos\beta = \frac{1}{2}(\sin(\alpha - \beta) + \sin(\alpha + \beta))$.

2. $\sin\alpha \cdot \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

3. $\cos\alpha - \cos\beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta)) = \frac{\sin(\alpha + \beta)}{tg\alpha + tg\beta}$.

4. $tg\alpha \cdot tg\beta = \frac{tg\alpha + tg\beta}{ctg\alpha + ctg\beta} = \frac{tg\alpha - tg\beta}{ctg\beta - ctg\alpha}$.

5. $ctg\alpha \cdot ctg\beta = \frac{ctg\alpha + ctg\beta}{tg\alpha + tg\beta} = \frac{ctg\alpha - ctg\beta}{tg\beta - tg\alpha}$.

6. $tg\alpha \cdot ctg\beta = \frac{tg\alpha + ctg\beta}{ctg\alpha + tg\beta} = \frac{tg\alpha - ctg\beta}{tg\beta - ctg\alpha}$.

7. $\sin(\alpha + \beta) \cdot \sin(\alpha - \beta) = \cos^2\beta - \cos^2\alpha$.

8. $\cos(\alpha + \beta) \cdot \sin(\alpha - \beta) = \sin\alpha\ \cos\alpha - \sin\beta \cos\beta$.

9. $\cos(\alpha + \beta) \cdot \cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$.

10. $\sin(\alpha + \beta) \cdot \cos(\alpha - \beta) = \sin\alpha\ \cos\alpha + \sin\beta \cos\beta$.

BA‘ZI TRIGONOMETRIK FORMULALAR.

1. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha$.

2. $\cos 3\alpha = \cos^3\alpha - 3\cos\alpha \sin^2\alpha = 4\cos^3\alpha - 3\cos\alpha$.

3. $tg 3\alpha = \frac{tg\alpha + tg 2\alpha}{1 - tg\alpha . tg 2\alpha} = \frac{3tg\alpha - tg^3\alpha}{1 - 3tg^2\alpha}$.

4. $\sin 4\alpha = 4\sin\alpha \cos\alpha\ (1 - 2\sin^2\alpha)$

5. $\cos 4\alpha = 8\sin^4\alpha - 8\sin^2\alpha + 1$

6. $tg 4\alpha = \frac{4tg\alpha(1 - tg^2\alpha)}{1 - 6tg^2\alpha + tg^4\alpha}$.

7. $ctg 4\alpha = \frac{4tg\alpha(1 - tg^2\alpha)}{tg^4\alpha - 6tg^2\alpha + 1}$.

8. $\sin 5\alpha = 16\sin\alpha\ \cos^4\alpha - 12\sin\alpha \cos^2\alpha + \sin\alpha$.

9. $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$.

10. $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha = \dfrac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}, \ \alpha \neq 2k\pi,\ k \in Z.$

11. $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha = \dfrac{\sin\frac{n\alpha}{2}\sin\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}, \ \alpha \neq 2k\pi,\ k \in Z.$

12. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) = \ = \sin\left(\alpha + \dfrac{(n-1)\beta}{2}\right)\sin\dfrac{n\beta}{2}.$

13. $\sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots + \sin^2 n\alpha = \dfrac{n+1}{2} - \dfrac{\cos n\alpha \sin(n+1)\alpha}{2\sin\alpha},$

$\alpha \neq 2k\pi,\ k \in Z.$

14. $\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos 2^n\alpha = \dfrac{1}{2^{n+1}} \dfrac{\sin 2^{n+1}\alpha}{\sin\alpha}, \ \alpha \neq 2k\pi,\ k \in Z.$

15. $a\sin\alpha \pm b\cos\beta = \sqrt{a^2+b^2}\sin(\alpha \pm \beta)$, bu yerda $\sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$,

$\cos\beta = \dfrac{b}{\sqrt{a^2+b^2}}.$

16. $\max(a\sin\alpha \pm b\cos\alpha) = \sqrt{a^2+b^2}.$

17. $\min(a\sin\alpha \pm b\cos\alpha) = -\sqrt{a^2+b^2}.$

TRIGONOMETRIK FUNKSIYALARNING ISHORALARI.

α	I chorak	II chorak	III chorak	IV chorak
$\sin\alpha$	+	+	-	-
$\cos\alpha$	+	-	-	+
$\mathrm{tg}\,\alpha$	+	-	+	-
$\mathrm{ctg}\,\alpha$	+	-	+	-

TRIGONOMETRIK FUNKSIYALARNING BA'ZI XOSSALARI.

Son argumentining sinusi, kosinusi, tangensi, kotangensi trigonometrik funksiyalar deyilib, ular mos ravishda *y=sin x, y= cos x, y= tgx, y=ctgx* ko'rinishida yoziladi.

1. *sinx, tgx, ctgx* - toq funksiyalar, cos*x*-juft funksiya, ya'ni *sin(-x)= -sinx, tg(-x)=-tgx, ctg(-x)= – ctgx, cos(-x)=cosx.*

2. *sinx, cosx* - davriy funksiyalar bo'lib, eng kichik musbat davri $T=2\pi$ ga teng, *tgx, ctgx* - davriy funksiyalar bo'lib, eng kichik musbat davri $T=\pi$ ga teng.

3. *y=sin x* funksiyaning aniqlanish sohasi barcha haqiqiy sonlar to'plamidan iborat: D(x)= R. Qiymatlar sohasi esa [-1;1] kesmadan iborat: E(y)=

[-1;1]. Ushbu funksiya $\left[-\frac{\pi}{2}+2\pi n;\frac{\pi}{2}+2\pi n\right]$, $(n\in Z)$ oraliqda o‘suvchi; $\left[\frac{\pi}{2}+2\pi n;\frac{3\pi}{2}+2\pi n\right]$, $(n\in Z)$ oraliqda kamayuvchidir.

4. *y= cos x* funksiyaning aniqlanish sohasi barcha haqiqiy sonlar to‘plamidan iborat: *D(x)*= R. Qiymatlar sohasi esa [-1;1] kesmadan iborat: E(y)= [-1;1]. Ushbu funksiya $[2\pi n,\pi+2\pi n]$,$(n\in Z)$ oraliqda kamayuvchi, $[\pi+2\pi n;2\pi+2\pi n]$, $(n\in Z)$ oraliqda o‘suvchidir.

5. *y= tgx* funksiyaning aniqlanish sohasi *D(x)*=$\left(-\frac{\pi}{2}+\pi n;\frac{\pi}{2}+\pi n\right)$, $(n\in Z)$; qiymatlar sohasi esa *E(y)= R*. *y= tgx* funksiya har bir $\left(-\frac{\pi}{2}+\pi n;\frac{\pi}{2}+\pi n\right)$, $(n\in Z)$ oraliqda $-\infty$ dan $+\infty$ gacha o‘suvchi.

6. *y=ctgx* funksiyaning aniqlanish sohasi *D(x)*=$(\pi n;\pi+\pi n)$, $(n\in Z)$; qiymatlar sohasi esa *E(y)= R*. *y= ctgx* funksiya har bir $(\pi n;\pi+\pi n)$, $(n\in Z)$ oraliqda $+\infty$ dan $-\infty$ gacha kamayuvchi.

TESKARI TRIGONOMETRIK FUNKSIYALAR

y=arcsinx funksiya: 1) *D*(*x*)=[-1;1];

2) *E*(*y*)= $\left[\frac{-\pi}{2};\frac{\pi}{2}\right]$;

3) arcsin(-*x*)=-arcsin*x* tenglik bajarilgani uchun bu funksiya toqdir;

4) (0;1]oraliqda musbat, [-1;0) oraliqda manfiy qiymatlidir;

5) [-1;1] kesmada o‘suvchi bo‘lib, bu oraliqning chap oxirida o‘zining eng kichik $-\frac{\pi}{2}$ qiymatiga, o‘ng oxirida eng katta $\frac{\pi}{2}$ qiymatiga erishadi.

Bu funksiyaning grafigi *y=sinx* funksiyanig $\left[\frac{-\pi}{2};\frac{\pi}{2}\right]$ oraliqdagi grafigini *y=x* to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi.

y=arccosx funksiya: 1) *D*(*x*)=[-1;1];

2) *E*(*y*)= $[0;\pi]$;

3) Funksiya toq ham, juft ham emas;

4) [-1;1]oraliqda musbat qiymatlidir;

5) [-1;1] kesmada kamayuvchi bo‘lib, bu oraliqning chap oxirida o‘zining eng katta π qiymatiga, o‘ng oxirida eng kichik 0 qiymatiga erishadi.

Bu funksiyaning grafigi *y=cosx* funksiyanig $[0;\pi]$ oraliqdagi grafigini *y=x* to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi.

y=arctgx funksiya: 1) *D*(*x*)=R;

2) *E*(*y*)= $\left(\frac{-\pi}{2};\frac{\pi}{2}\right)$;

3) arctg(-x)=-arctg(x) tenglik bajarilgani uchun bu funksiya toqdir;

4) R_+ oraliqda musbat, R_- oraliqda manfiy qiymatlidir;

5) *R* haqiqiy sonlar to‘plamida o‘suvchi.

Bu funksiyaning grafigi $y=tgx$ funksiyanig $\left(\frac{-\pi}{2};\frac{\pi}{2}\right)$ oraliqdagi grafigini $y=x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi.

y=arcctgx funksiya: 1) $D(x)=R$;

2) $E(y)=(0;\pi)$;

3) Funksiya toq ham, juft ham emas;

4) R to‘plamning har bir nuqtasida musbat qiymatlidir;

5) R to‘plamda kamayuvchi.

Bu funksiyaning grafigi $y=ctgx$ funksiyanig $(0;\pi)$ oraliqdagi grafigini $y=x$ to‘g‘ri chizig‘iga nisbatan simmetrik yasash bilan hosil qilinadi.

Teskari trigonometrik funksiyalar uchun quyidagi munosabatlar o‘rinli:

1. $\arcsin(\sin x)=x,\ x\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$; 2. $\sin(\arcsin x)=x,\ x\in[-1;1]$;

3. $\arccos(\cos x)=x,\ x\in[0;\pi]$; 4. $\cos(\arccos x)=x,\ x\in[-1;1]$;

5. $\mathrm{arctg}(\mathrm{tg}x)=x,\ x\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$; 6. $\mathrm{tg}(\mathrm{arctg}x)=x,\ x\in R$;

7. $\mathrm{arcctg}(\mathrm{ctg}x)=x,\ x\in(0;\pi)$; 8. $\mathrm{ctg}(\mathrm{arcctg}x)=x,\ x\in R$;

9. $\arcsin(-x)=-\arcsin x$; 10. $\mathrm{acrcos}(-x)=\pi-\arccos x$;

11. $\mathrm{arctg}(-x)=-\mathrm{arctg}x$; 12. $\mathrm{arcctg}(-x)=\pi-\mathrm{arcctg}x$;

13. $\arcsin x=\frac{\pi}{2}-\arccos x=\mathrm{arctg}\frac{x}{\sqrt{1-x^2}}$;

14. $\arccos x=\frac{\pi}{2}-\arcsin x=\mathrm{arcctg}\frac{x}{\sqrt{1-x^2}}$;

15. $\mathrm{arctg}x=\frac{\pi}{2}-\mathrm{arcctg}x=\arcsin\frac{x}{\sqrt{1+x^2}}$;

16. $\mathrm{arcctg}x=\frac{\pi}{2}-\mathrm{arctg}x=\arccos\frac{x}{\sqrt{1+x^2}}$;

17. $\sin(\arccos x)=\sqrt{1-x^2}$; 18. $\sin(\mathrm{arctg}x)=\frac{x}{\sqrt{1+x^2}}$;

19. $\sin(\mathrm{arcctg}x)=\frac{1}{\sqrt{1+x^2}}$; 20. $\sin(2\arcsin x)=2x\sqrt{1-x^2}$;

21. $\sin(2\arccos x)=2x\sqrt{1-x^2}$; 22. $\sin(2\mathrm{arctg}x)=\frac{2x}{1+x^2}$;

23. $\sin(2\mathrm{arcctg}x)=\frac{2x}{1+x^2}$; 24. $\cos(\arcsin x)=\sqrt{1-x^2}$;

25. $\cos(\mathrm{arctg}x)=\frac{1}{\sqrt{1+x^2}}$; 26. $\cos(\mathrm{arcctg}x)=\frac{x}{\sqrt{1+x^2}}$;

27. $\cos(2\arcsin x)= -(2x^2-1)$; 28. $\cos(2\arccos x)=2x^2-1$;

29. $\cos(2\mathrm{arctg}x)= \frac{1-x^2}{1+x^2}$; 30. $\cos(2\mathrm{arcctg}x)=\frac{x^2-1}{x^2+1}$;

31. $\mathrm{tg}(\arcsin x)=\frac{x}{\sqrt{1-x^2}}$; 32. $\mathrm{tg}(\arccos x)=\frac{\sqrt{1-x^2}}{x}$;

33. $\mathrm{tg}(\mathrm{arcctg}x)=\frac{\sqrt{1+x^2}}{x^2-1}$; 34. $\mathrm{tg}(2\arcsin x)=\frac{2x\sqrt{1-x^2}}{1-2x^2}$;

35. $\mathrm{tg}(2\arccos x)=\frac{2x\sqrt{1-x^2}}{2x^2-1}$; 36. $\mathrm{tg}(2\mathrm{arctg}x)=\frac{2x}{1-x^2}$;

37. $\mathrm{tg}(2\mathrm{arcctg}x)=\frac{2x}{x^2-1}$; 38. $\mathrm{ctg}(\arcsin x)=\frac{\sqrt{1-x^2}}{x}$;

39. $\mathrm{ctg}(2\arcsin x)=\frac{x}{\sqrt{1-x^2}}$; 40. $\mathrm{ctg}(\mathrm{arctg}x)=\frac{1}{x}$;

41. $\mathrm{ctg}(2\arcsin x)=\frac{1-2x^2}{2x\sqrt{1-x^2}}$; 42. $\mathrm{ctg}(2\arccos x)=\frac{2x^2-1}{2x\sqrt{1-x^2}}$;

43. $\mathrm{ctg}(2\mathrm{arctg}x)=\frac{1-x^2}{2x}$; 44. $\mathrm{ctg}(2\mathrm{arcctg}x)=\frac{x^2-1}{2x}$;

45. $\arccos x+\arcsin x=\pi/2$; 46. $\mathrm{arctg}x+\mathrm{arcctg}x=\pi/2$;

47. $\arccos x+\arccos y=-\arccos\left(xy-\sqrt{1-x^2}\cdot\sqrt{1-y^2}\right),\ x\geq y$;

48. $\arccos x+\arccos y=\arccos\left(xy-\sqrt{1-x^2}\cdot\sqrt{1-y^2}\right),\ x<y$;

49. $\mathrm{arctg}x+\mathrm{arctg}y=\mathrm{arctg}\frac{x+y}{1-xy},\ xy<1$; 50. $\mathrm{arctg}x-\mathrm{arctg}y=\mathrm{arctg}\frac{x-y}{1-xy},\ xy>1$

51. $\mathrm{arcctg}x+\mathrm{arcctg}y=\mathrm{arcctg}\frac{xy-1}{x+y},\ x\neq -y$;

52. $\mathrm{arcctg}x-\mathrm{arcctg}y=\mathrm{arcctg}\frac{xy+1}{x-y},\ x\neq y$.

TRIGONOMETRIK TENGLAMALAR

1. $\sin x=a,\ |a|\leq 1,\ \ x=(-1)^n\arcsin a+\pi n,\ n\in Z$;
2. $\cos x=a,\ |a|\leq 1,\ \ x=\pm\arccos a+2\pi n,\ n\in Z$;
3. $\mathrm{tg}x=a,\ x=\mathrm{arctg}a+\pi n\ \ n\in Z$;
4. $\mathrm{ctg}x=a,\ x=\mathrm{arcctg}a+\pi n,\ \ n\in Z$.

XUSUSIY HOLLAR

a	$\sin x=a$	$\cos x=a$
0	$x=\pi k,\ k\in Z.$	$x=\frac{\pi}{2}+\pi k,\ k\in Z.$
1	$x=\frac{\pi}{2}+2\pi k,\ k\in Z.$	$x=2\pi k,\ k\in Z.$
-1	$x=-\frac{\pi}{2}+2\pi k,\ k\in Z.$	$x=\pi+2\pi k,\ k\in Z.$
$\frac{1}{2}$	$x=(-1)^n\frac{\pi}{6}+\pi k,\ k\in Z.$	$x=\pm\frac{\pi}{3}+2\pi k,\ k\in Z.$
$-\frac{1}{2}$	$x=(-1)^{n+1}\frac{\pi}{6}+\pi k,\ k\in Z.$	$x=\pm\frac{2\pi}{3}+2\pi k,\ k\in Z.$
$\frac{\sqrt{3}}{2}$	$x=(-1)^n\frac{\pi}{3}+\pi k,\ k\in Z.$	$x=\pm\frac{\pi}{6}+2\pi k,\ k\in Z.$
$-\frac{\sqrt{3}}{2}$	$x=(-1)^{n+1}\frac{\pi}{3}+\pi k,\ k\in Z.$	$x=\pm\frac{5\pi}{6}+2\pi k,\ k\in Z.$
$\frac{\sqrt{2}}{2}$	$x=(-1)^n\frac{\pi}{4}+\pi k,\ k\in Z.$	$x=\pm\frac{\pi}{4}+2\pi k,\ k\in Z.$
$-\frac{\sqrt{2}}{2}$	$x=(-1)^{n+1}\frac{\pi}{4}+\pi k,\ k\in Z.$	$x=\pm\frac{3\pi}{4}+2\pi k,\ k\in Z.$
$a>1, a<-1$	$\varnothing$	$\varnothing$

a	$\mathrm{tg}x$	$\mathrm{ctg}x$
0	$x=\pi n,\ n\in Z.$	$x=\frac{\pi}{2}+\pi n,\ n\in Z.$
1	$x=\frac{\pi}{4}+\pi n,\ n\in Z.$	$x=\frac{\pi}{4}+\pi n,\ n\in Z.$
-1	$x=-\frac{\pi}{4}+\pi n,\ n\in Z.$	$x=\frac{3\pi}{4}+\pi n,\ n\in Z.$
$\sqrt{3}$	$x=\frac{\pi}{3}+\pi n,\ n\in Z.$	$x=\frac{\pi}{6}+\pi n,\ n\in Z.$
$-\sqrt{3}$	$x=-\frac{\pi}{3}+\pi n,\ n\in Z.$	$x=\frac{5\pi}{6}+\pi n,\ n\in Z.$
$\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$	$x=\frac{\pi}{6}+\pi n,\ n\in Z.$	$x=\frac{\pi}{3}+\pi n,\ n\in Z.$
$-\frac{\sqrt{3}}{3}=-\frac{1}{\sqrt{3}}$	$x=-\frac{\pi}{6}+\pi n,\ n\in Z.$	$x=\frac{2\pi}{3}+\pi n,\ n\in Z.$

KELTIRISH FORMULALARI.

x	$\sin x$	$\cos x$	$\text{tg } x$	$\text{ctg } x$
$\frac{\pi}{2}-\alpha$	$\cos\alpha$	$\sin\alpha$	$\text{ctg }\alpha$	$\text{tg }\alpha$
$\frac{\pi}{2}+\alpha$	$\cos\alpha$	$-\sin\alpha$	$-\text{ctg }\alpha$	$-\text{tg }\alpha$
$\pi-\alpha$	$\sin\alpha$	$-\cos\alpha$	$-\text{tg }\alpha$	$-\text{ctg }\alpha$
$\pi+\alpha$	$-\sin\alpha$	$-\cos\alpha$	$\text{tg }\alpha$	$\text{ctg }\alpha$
$\frac{3\pi}{2}-\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\text{ctg }\alpha$	$\text{tg }\alpha$
$\frac{3\pi}{2}+\alpha$	$-\cos\alpha$	$\sin\alpha$	$-\text{ctg }\alpha$	$-\text{tg }\alpha$
$2\pi-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\text{tg }\alpha$	$-\text{ctg }\alpha$
$2\pi+\alpha$	$\sin\alpha$	$\cos\alpha$	$\text{tg }\alpha$	$\text{ctg }\alpha$

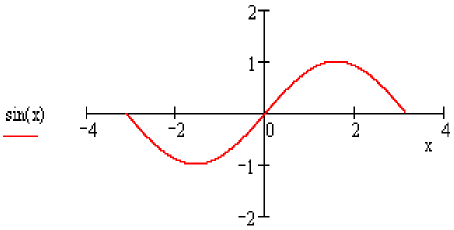
BA'ZI BURCHAKLARNING TRIGONOMETRIK FUNKSIYALARDAGI QIYMATLARI

α^0	α radian	$\sin\alpha$	$\cos\alpha$	$tg\alpha$	$ctg\alpha$
15^0	$\frac{\pi}{12}$	$\frac{\sqrt{2-\sqrt{3}}}{2}$	$\frac{\sqrt{2+\sqrt{3}}}{2}$	$2-\sqrt{3}$	$2+\sqrt{3}$
22.5^0	$\frac{\pi}{8}$	$\sqrt{\frac{2-\sqrt{2}}{2}}$	$\sqrt{\frac{2+\sqrt{2}}{2}}$	$\sqrt{2}-1$	$\sqrt{2}+1$
30^0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45^0	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60^0	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
75^0	$\frac{5\pi}{12}$	$\frac{\sqrt{2+\sqrt{3}}}{2}$	$\frac{\sqrt{2-\sqrt{3}}}{2}$	$2+\sqrt{3}$	$2-\sqrt{3}$
90^0	$\frac{\pi}{2}$	1	0	-	0
105^0	$\frac{7\pi}{12}$	$\frac{\sqrt{2+\sqrt{3}}}{2}$	$-\frac{\sqrt{2-\sqrt{3}}}{2}$	$-(2+\sqrt{3})$	$\sqrt{3}-2$
120^0	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$
135^0	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1

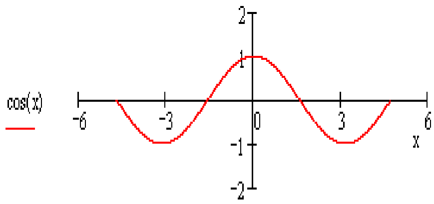
150^0	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$
165^0	$\frac{11\pi}{12}$	$\frac{\sqrt{2-\sqrt{3}}}{2}$	$-\frac{\sqrt{2+\sqrt{3}}}{2}$	$\sqrt{3}-2$	$-(2+\sqrt{3})$
180^0	π	0	-1	0	-
210^0	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	$\sqrt{3}$
240^0	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
270^0	$\frac{3\pi}{2}$	-1	0	-	0
300^0	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$
330^0	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$
0^0, 360^0	2π	0	1	0	-

TRIGONOMETRIK FUNKSIYALAR GRAFIKLARI

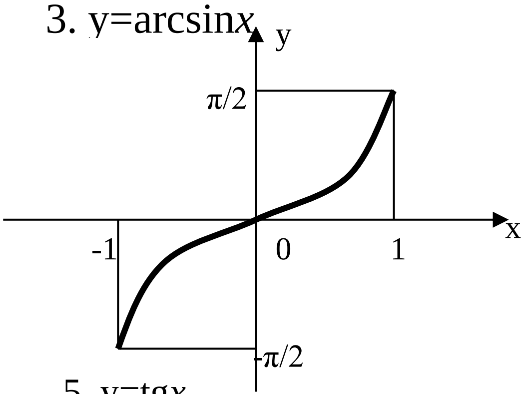
1. y=sin*x*



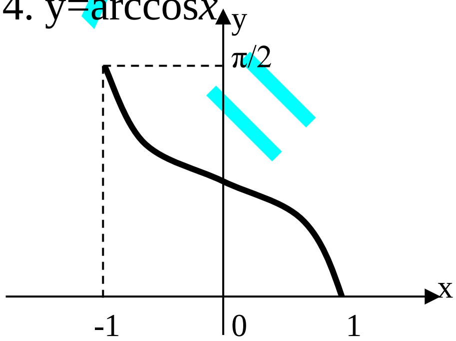
2. y=cos*x*



3. y=arcsin*x*

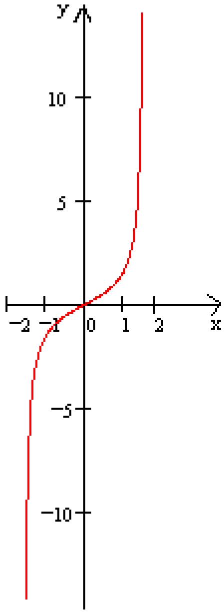


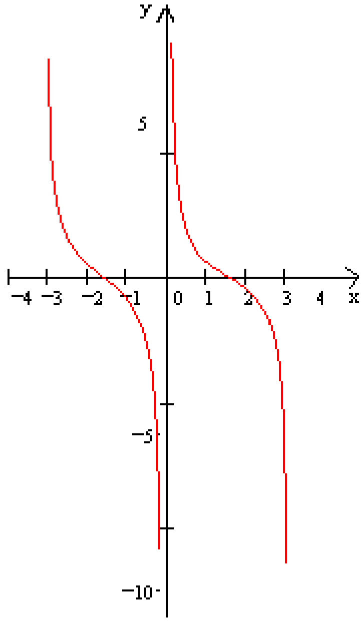
4. y=arccos*x*



5. y=tg*x*

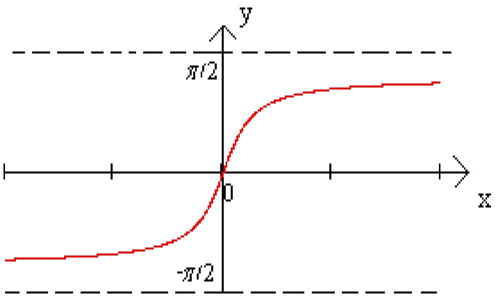
6. y=ctg*x*

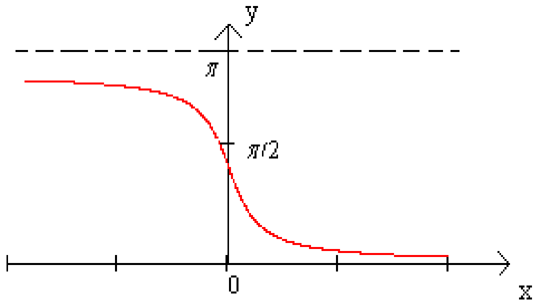




7. y=arctg$x$

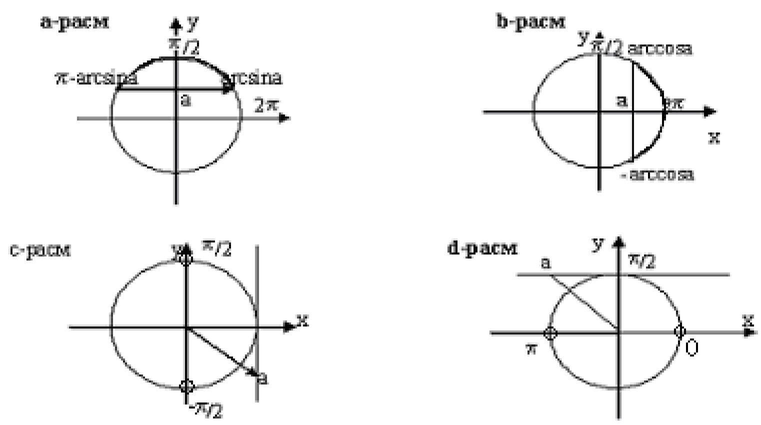
8. y=arcctg$x$





TRIGONOMETRIK TENGSIZLIKLAR.

1. sin$x$ $>a$, $|a|<1$ $\quad x\in$ $(\arcsin a+2\pi n;\pi-\arcsin a+2\pi n), n\in Z$. (a-rasm)
2. sin$x$ $<a$, $|a|<1$ $\quad x\in(-\pi-\arcsin a+2\pi n;\arcsin a+2\pi n), n\in Z$.
3. cos$x$>a, $|a|<1$ $\quad x\in(-\arccos a+2\pi n;\arccos a+2\pi n)$. (b-rasm)
4. cos$x$<a, $|a|<1$ $\quad x\in(\arccos a+2\pi n;2\pi-\arccos a+2\pi n)$.
5. tg$x$>a $x\in(arctga+\pi n;\pi/2+\pi n)$. (c-rasm)
6. tg$x$<a, $x\in(-\pi/2+\pi n;arctga+\pi n)$.
7. ctg$x$>a, $x\in(\pi n;arcctga+\pi n)$. (d-rasm)
8. ctg$x$<a, $x\in(arcctga+\pi n;\pi+\pi n), n\in Z$.



HOSILANING MEXANIK MA'NOSI

Agar $S(t)$ - masofa (yo'l), $V(t)$ - tezlik, $a(t)$ - tezlanish (oniy tezlik) bo'lsa, $S'(t)=V(t)$, $S''(t)=V'(t)=a(t)$ bo'ladi.

Egri chiziqlar orasidagi burchak

$M(x_0, y_0)$ nuqtada kesishuvchi $f_1(x)$ va $f_2(x)$ egri chiziqlar orasidagi burchak:

$$tg\alpha = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)}.$$

AYRIM FUNKSIYANING ENG KATTA VA ENG KICHIK QIYMATLARI.

1. $y = ax^2 + bx + c$

Parabola uchi koordinatalari: $x_0 = -\frac{b}{2a}$; $y_0 = -\frac{b^2 - 4ac}{4a} = ax_0^{'} + bx_0 + c$.

Agar a>0 bo'lsa, $\min y = y_0$ bo'ladi.

Agar a<0 bo'lsa, $\max y = y_0$ bo'ladi.

2. $y = \sin kx$ va $y = \cos kx$ funksiyalar uchun $\max y = 1$, $\min y = -1$.

3. $y = a\sin kx + b\cos kx$ funksiya uchun $\max y = \sqrt{a^2 + b^2}$, $\min y = -\sqrt{a^2 + b^2}$.

AYRIM MASALALARNI YECHISH UCHUN FORMULALAR.

1. Xaritada masofa *A* sm bo'lsa, yer yuzida qancha (*x*) bo'ladi, M(*1:B*) – mashtab.

$$x = \frac{A \cdot B}{100000} km = \frac{A \cdot B}{100} m = A \cdot B sm = 10A \cdot B mm.$$

2. Yer yuzida *A km* bo'lsa, xaritada qancha (*y*) bo'ladi, M(*1:B*) – mashtab.

$$y = \frac{100000A}{B} sm = \frac{10000A}{B} dm = \frac{1000000A}{B} mm.$$

3. 1 dan C gacha bo'lgan sonlar orasidan $<A>$ va $<B>$ ga bo'linadigan (D) nechta (A va B o'zaro tub sonlar) son bor?

$$D = C - \left[\frac{C}{A}\right] - \left[\frac{C}{B}\right] + \left[\frac{C}{A \cdot B}\right].$$

4. Bir ishchi ishning *A* qismini (t_1 vaqtda), ikkinchisi shu ishning *B* qismini (t_2 vaqtda) bajarsa, ikkalasi shu ishni qanchasini (*t* vaqtda) (*X*) bajaradi?

$$\frac{t_1}{A}+\frac{t_2}{B}=\frac{t}{X}.$$

5. Jami o‘quvchi – A ta, ingliz tilini biladiganlari – B ta, nemis tilini biladiganlari – C ta, ikki tilni biladiganlari – D ta, ikki tilni bilamaydiganlari – E ta

$$E=(A+D)-(B+C).$$

Ushbu formulaning isboti: X=B-D –faqat ingliz tilini biladigan o‘quvchilar soni; Y=C-D –faqat nemis tilini biladigan o‘quvchilar soni; X+Y+D+E=A. X va Y ning qiymatlarini keltirib qo‘yamiz:
B-D+C-D+D+E=A va natijada yuqoridagi $E=(A+D)-(B+C)$ formula kelib chiqadi.

6. Jami o‘quvchi – A ta, raqs to‘garagida – B ta, ashula to‘garagida – C ta, faqat raqs to‘garagida – X ta, faqat ashula to‘garagida – Y ta, ikkalasida ham – Z ta

$$X=A-C \qquad Y=A-B \qquad Z=(B+C)-A$$

7. Agar savdogar mahsulotining 1 kg ni A so‘mdan sotsa, B so‘m zarar ko‘radi. Agar C so‘mdan sotsa, D so‘m foyda ko‘radi. Savdogarning mahsuloti (X) qancha?

$$X=\frac{B+D}{C-A}$$

8. Tarkibida P% oltin bo‘lgan A gr modda tarkibida Q% oltin bo‘lgan B gr modda bilan aralashtirildi:
1) Birinchi moddadagi sof oltin miqdori (x)?

$$x=\frac{A\cdot P}{100}$$

2) Ikkinchi moddadagi sof oltin miqdori (x)?

$$x=\frac{B\cdot Q}{100}$$

3) Aralashmaning massasi (x)?

$$x=A+B$$

4) Aralashmadagi sof oltin miqdori (x)?

$$x=\frac{A\cdot P+B\cdot Q}{A+B}$$

9. n! nechta nol bilan tugaydi?

$$x=\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\left[\frac{n}{5^4}\right]+\left[\frac{n}{5^5}\right]+\ldots+\left[\frac{n}{5^k}\right],\ 5^{K+1}>n.$$

[a] – a dan oshmaydigan eng katta butun son.

SONLARGA DOIR MASALALAR

1-M. 345678910111213…686970 sonlarning raqamlar yig’indisini toping. Sonni to’liqroq ko’rinishga keltiramiz.
12345678910111213…678970
123456789 uchun 1+2+3+…+9=45
10111213…19 uchun 55
20212223…29 uchun 65

30313233…39 uchun 75
40414243…49 uchun 85
50515253…59 uchun 95
60616263…69 uchun 105
70 uchun 7+0=7
12 uchun 1+2=3
Jami: 45+55+65+75+85+95+105+7-3=529

2-M. $1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \ldots \cdot n$ ko'paytmadagi nollar sonini toping.

$$\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\left[\frac{n}{5^4}\right]+\ldots$$

Maxraj suratdan (n dan) katta bo'lguncha davom ettiriladi va barcha qavslarning butun qismlari qo'shib chiqiladi. Butun qismga masalan: $[2{,}154]=2$.

3-M. 1 dan N gacha natural sonlar ichida a ga ham b ga ham bo'linmaydiganlarining soni

$$N+\left[\frac{N}{a\cdot b}\right]-\left[\frac{N}{a}\right]-\left[\frac{N}{b}\right]$$

4-M. a ga karrali N dan katta bo'lmagan barcha natural sonlar yig'inisini topish.

$$\frac{a\left(1+\left[\frac{N}{a}\right]\right)}{2}\cdot\left[\frac{N}{a}\right]$$

5-M. bir necha natural sonlarning yig'indisi a ga teng. Agar ularning har biridan b ni ayirib (har biri b ga orttirib) yig'indi hisoblansa, u c ga teng bo'ladi. Yig'indida nechta natural son bor?

$$a-bx=c \quad (a+bx=c)$$

6-M. ko'paytmaning har bir hadi a ga ko'paytirildi (a ga bo'lindi). Natijada ko'paytma b marta ortdi (b marta kamaydi). Ko'paytmada nechta had qatnashgan?

$$a^r=b$$

7-M. kitob betlarini sahifalash. Agar kitobning oxirgi sahifasi uch xonali son bilan tugagan bo'lsa? N -jami ishlatilgan raqamlar soni; x -kitobning necha betligi;

$$x=\frac{N+110}{3}$$

8-M. biror sonning o'ng tomoniga 0 raqami yozish (yoki 0 raqamini o'chirish) bu sonni 10 ga ko'paytirish (10 ga bo'lish) degani bo'ladi.
Ya'ni: Biror x sonning o'ng tomoniga 0 raqami yozilganda $10x$ bo'ladi.
Biror x sonning o'ng tomonidan 0 raqami o'chirilganda $\frac{x}{10}$ bo'ladi.

9-M. Ikki xonali son. $\overline{xy}=10x+y$. x -o'nliklar xonasidagi raqam; y -birliklar xonasidagi raqam;
a) Raqamlarning yig'inidisi; $x+y$ (ayirmasi $x-y$);
b) Raqamlarning ko'paytmasi; xy (bo'linmasi $\frac{x}{y}$);

c) $\overline{xy} = 10x + y$ ikki xonali sonning raqamlari o'rni almashsa: $\overline{yx} = 10y + x$.

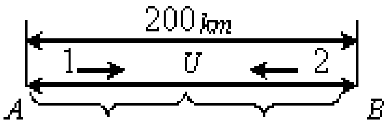
10-M. Jami N ta turistdan a tasi ingliz tili, b tasi nemis, c tasi ikkala tilni bilsa, d tasi esa ikkala tilni ham bilmasa, quyidagi munosabat o'rinli bo'ladi:

$$N = a + b + d - c$$

HARAKATGA DOIR MASALALAR

Formulalar, ϑ-tezlik, S-masofa, t-vaqt $S = \vartheta t$, $\vartheta = \frac{S}{t}$, $t = \frac{S}{\vartheta}$.

1-M. Orasidagi masofa 200 km bo'lgan A va B punktlardan bir-biriga qarab ikki turist bir vaqtning o'zida yo'lga chiqdi. Birinchisi avtobusda tezligi 40 $km/soat$, ikkinchisi avtomobilda. Agar ular 2 soatdan keyin uchrashsa, avtomobilning tezligini toping.



Yechilishi:

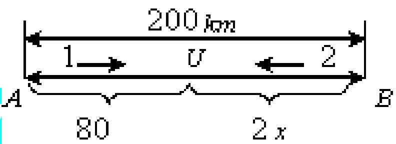
U nuqta uchrashish joyini bildiradi. Quyidagi jadvalga ma'lumotlarni joylashtirib chiqamiz:

	ϑ, $\frac{km}{soat}$	t, soat	S, km
1-turist	40	2	
2-turist		2	

$S = \vartheta t$ formuldan foydalanib jadvalni to'ldiramiz.

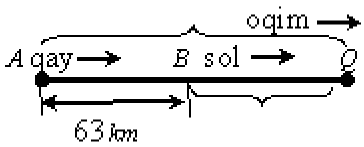
	ϑ, $\frac{km}{soat}$	t, soat	S, km
1-turist	40	2	80
2-turist	x	2	$2x$

Rasmda faqat jadvaldagi masofalarni olib borib ko'rsatamiz.



Demak, $80 + 2x = 200 \Rightarrow x = 60\ km/soat$.

2-M. Ikki pristan orasidagi masofa 63 km. Bir vaqtning o'zida oqim bo'ylab birinchi pristandan sol, ikkinchisidan motorli qayiq jo'natildi va motorli qayiq solni 3 soatda quvib yeti. Agar daryo oqimining tezligi 3 $km/soat$ bo'lsa, qayiqning turg'un suvdagi tezligini toping.



Yechilishi: Q-qayiqning solni quvib yetish joyi;

AQ-qayiqning solni quvib yetguncha boib o'tgan masofasi; BQ-solning (yog'ochning) masofasi.

Tezliklarni mulohaza qilamiz:

oqimning tezligi(o.t)–3 $km/soat$;

qayiqning turg'un (oqmaydigan) suvdagi tezligi (q.t)– $x\ km/soat$;

oqim bo'yicha tezlik (o.b.t)– $x + 3$;

oqimga qarshi tezlik– $x - 3$;

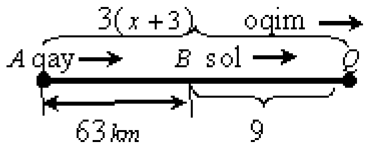
Quyidagi jadvalni to'ldiramiz:

	ϑ, $\frac{km}{soat}$	t, soat	S, km

qayiq	$x+3$	3	$3(x+3)$
sol	3	3	9

Jadvaldagi masofalarni rasmga olib borib ko'rsatamiz:

$3(x+3)=63+9 \Rightarrow x=21\,km/soat$



3-M. Ikki shahardan bir vaqtning o'zida turli tezlik bilan ikki avtomobil bir-biriga qarab yo'lga chiqdi. Avtomobillarning har biri uchrashish joyigacha bo'lgan masofaning yarmini bosib o'rgandan keyin, haydovchilar tezlikni 1,5 baravar oshirdi, natijada avtomobillar belgilangan muddatdan 1 soat oldin uchrashishdi. Harakat boshlangandan necha soatdan keyin avtomobillar uchrashishdi?

Yechilishi: Belgilangan vaqt x bo'lsin.

1-hol. Avtomobillarning har biri uchrashish joyigacha bo'lgan masofaning yarmini bosib o'tadi. Demak, vaqt masofaga tog'ri prororsional bo'lgani uchun mo'ljaldagi vaqtning ham yarmi ketadi, yani $\frac{x}{2}$ soat ketadi, $\frac{x}{2}$ soat qoladi.

2-hol. Yo'lning qolgan qismida har bir hydovchi o'z tezligini 1,5 marta ohiradi. Vaqt tezlikka teskari proporsional bo'lgani uchun qolgan $\frac{x}{2}$ soat 1,5 marta kamayadi, ya'ni yo'lning qolgan qismi $\frac{\frac{x}{2}}{1,5}=\frac{x}{3}$ soat vaqt ketadi.

Haydovchilar uchrashguncha jami: $\frac{x}{2}+\frac{x}{3}=\frac{5x}{6}$ soat vaqt ketadi.

Haydovchilar belgilangan vaqtdan 1 soat oldin uchrashishgani uchun: $\frac{5x}{6}=x-1 \Rightarrow x=6$ soat.

Javob: Avtomobillar $x=6$ soardan so'ng uchrashishlari kerak bo'lgan va 6-1=5 soatdan so'ng uchrashishgan.

III. ISHGA DOIR MASALALAR

1-M. $\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}$ formulaga doir.

Meshdagi suv Anvarning o'ziga 35 kunga, ukasi ikkalasiga esa 10 kunga yetadi. Meshdagi suv Anvarning ukasiga necha kunga yetadi?

Yechilishi: $t_1=35,\quad t=10,\quad t_2=?;\quad \frac{1}{10}=\frac{1}{35}+\frac{1}{x}$ tenglamadn $x=14$ kun kelib chiqadi.

Ishga doir masalalarni yechishga yordam

A-butun (to'la) ish;

x-ishchining ish unumi, ya'niishchining 1 soatda (1 kunda va h.k) qancha ish bajarishi;
t-ishni bajarishga ketgan vaqt.

$t=\frac{A}{x},\quad A=x\cdot t,\quad x=\frac{A}{t}$

1-M. Ihchi har soatda 5 ta detal yasaydi. U 20 ta detalni qancha vaqtda yasaydi?
Yechilishi: $x=5, A=20, t=?$

$t=\frac{A}{x}=\frac{20}{5}=4$ soat. Javob: 4 soatda.

Ba'zi bildiruvlar

•Butun ishning 40%i- $0{,}4A$;

•Butun ishning $\frac{5}{7}$ qismi- $\frac{5}{7}A$

•Ishchining ish unumi- x, u butun ishning 605ini qancha vaqtda bajaradi?
Butun ish- A.

Butun ishning 60%i- $0{,}6A$, $\frac{0{,}6A}{x}$ vaqtda bajaradi.

• x birinchi ishchining ish unumi
y-ikkinchi ishchining ish unumi
t_1x-birinchi ishchining t_1 vaqtda qancha ish bajarishi;
t_2y-ikkinchi ishchining t_2 vaqtda qancha ish bajarishi;
$t(x+y)$-ikkala ishchi birgalikda t vaqtda qancha ish bajarishi
$\frac{A}{x}$-birinchi ishchining o'zi butun ishning qancha vaqtda bajarishi;
$\frac{A}{y}$-ikkinchi ishchining o'zi butun ishning qancha vaqtda bajarishi;;
$\frac{A}{x+y}$ -ikkala ishchining birgalikda butun ishni qancha vaqtda bajarishi;
$A-t_1x$-birinchi ishchining o'zi t_1 vaqt ishlagandan so'ng qolgan ish;
$\frac{A-t_2y}{x+y}$-ikkinchi ishchining o'zi t_2 vaqt ishlagandan so'ng qolgan ishni ikkala ishchi birgalikda qancha vaqtda bajarishi.

Ishga doir masalalar

1-M. Bir ishchi buyurtmani 6 soatda? Boshqasi esa 10 soatda bajaradi. Ular birgalikda 3 soat ishlaganlaridan keyin ishning qancha qismi bajarilmay qoladi?
Yechilishi: 1) $\frac{A}{x}=6\Rightarrow x=\frac{A}{6}$; 2) $\frac{A}{y}=10\Rightarrow y=\frac{A}{10}$;

$A-3(x+y)=A-3\left(\frac{A}{6}+\frac{A}{10}\right)=A-3\cdot\frac{4}{15}A=\frac{1}{5}A$. Javob: Butun ishning $\frac{1}{5}$ qismi bajarilmay qoladi.

2-M. Usta muayyan ishni 12 kunda, uning shogirdi esa 30kunda bajaradi. Agar 3 ta usta va 5 ta shogird birga ishlasalar, o'sha ishni necha kunda bajaradi?

Yechilishi: 1) $\frac{A}{x}=12 \Rightarrow x=\frac{A}{12}$; 2) $\frac{A}{y}=30 \Rightarrow y=\frac{A}{30}$; 3) $\frac{A}{x+y}=\frac{A}{3\cdot\frac{A}{12}+5\cdot\frac{A}{30}}=2{,}4$.

Javob: 2,4 kunda.

ARALASHMAGA DOIR MASALALAR

1-M. Qotishma mis va qo'rg'oshindan iborat. Qo'rg'oshinning 60%i mis bo'lib, mis qo'rg'oshindan 2 kg ko'p. Qotishmada qancha mis bor.

Yechilishi: Qotishmani biror idishda deb tasavvur qilamiz:

	Massa (kg)	Foiz
Jami		100%
Mis	$x+2$	60%
Qo'shg'oshin	x	

Idishda chala yozilgan jadvalni to'ldiramiz:

	Massa (kg)	Foiz
Jami		100%
Mis	$x+2$	60%
Qo'shg'oshin	x	40%

Proporsiya tuzish uchun ikki qator ma'lumotni olish yetarli bo'lgani uchun eng qulayini tanlab olamiz:

$x+2$ - 60%

x -40%

$x\cdot 60=(x+2)\cdot 40$; $x=4$ kg. Mis, $x+2=4+2=6$ kg.

2-M. Massasi 400 g va konsentratsiyasi 8% bo'lgan eritma massasi 600 g va konsentratsiyasi 13% bo'lgan eritma bilan aralashtirildi. Hosil bo'lgan aralashma konsentratsiyasini toping.

Yechilishi: 1-usul. Quyidagi ma'lumotlarni qo'llash mumkin. Massasi m_1 va konsentratsiyasi a % bo'lgan eritma massasi m_2 va konsentratsiyasi b% bo'lgan eritma bilan aralashtirilganda hosil bo'lgan yangi eritmaning necha foizi (x) topish.

$$x=\frac{a\cdot m_1+b\cdot m_2}{m_1+m_2}.$$

Bizning masalada: $m_1=400\ g, a=8\%$; $m_2=600\ g, b=13\%$;

$x=\frac{8\cdot 400+13\cdot 600}{400+600}=11\%$. Javob: 11% li aralashma hosil bo'ladi.

2-usul. 1-eritma solingan ishga bor ma'lumotlardan joylaymiz.

	Massa (kg)	Foiz
Jami	400	100%
Suvi		
Konsentratsiyasi		8%

Jadvalni konsentratsiya qatorini to'lirsa yetarli. 400 g ning 8%i$=0{,}08\cdot 400=32\ g$.

	Massa (kg)	Foiz
Jami	400	100%
Suvi		
Konsentratsiyasi	32	8%

2-eritma uchun ham xuddi shunday yo'l bilan jadval to'ldiriladi.

	Massa (kg)	Foiz
Jami	600	100%
Suvi		
Konsentratsiyasi	78	13%

Ikkala eritma bir idishga solinganda bu eritmalarda faqat massalar qo'shiladi, foizlar esa qo'shilmaydi

	Massa (kg)	Foiz
Jami	600+400 = =1000	100%
Suvi		
Konsentratsiyasi	78+32 = =110	13%

Proporsiya tuzamiz: 1000 – 100%
110 - x

$x \cdot 1000 = 110 \cdot 100\%, x = 11\%$. Javob: 11% li aralashma hosil bo'ladi.

GEOMETRIYA

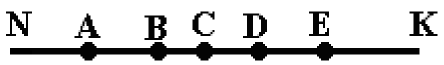
GEOMETRIYA – geometrik figuralar va ularning xossalarini o'rganuvchi fandir. “Geometriya” so'zi grekcha so'z bo'lib, o'zbekcha “yer o'lchash” degan ma'noni bildiradi.

Asosiy geometrik tudhunchalar nuqta, to'g'ri chiziq va tekislik hisoblanadi. Geometrik figura deb, nuqtalarning har qanday to'plamiga aytiladi. Agar geometrik figuraning barcha nuqtalar to'plami bir tekislikka tegishli bo'lsa, uni tekis geometrik figura deyiladi. Odatda nuqtalar lotin alfavitining bosh harflari bilan belgilanadi: A,B,C,D,… , to'gri chiziqlar esa lotin alfavitining kichik harflari bilan : a,b,c,d,…, yoki ikkita nuqta bilan belgilanadi: AB, AC, AD,…. Tekisliklar esa $\alpha, \beta, \gamma, ...$ kabi kichik grekcha harflar bilan ifodalanadi.

AKSIOMALAR:

1. Har qanday to'g'ri chiziq uchun unga tegishli va tegishli bo'lmagan nuqtalar mavjud.
2. Har qanday ikki nuqtadan yagona to'g'ri chiziq o'tkazish mumkin.
3. To'g'ri chiziqdagi uchta nuqtadan bittasi va faqat bittasi qolgan ikkitasi orasida yotadi.
4. Har bir kesma 0 dan katta uzunlikka ega.
5. To'g'ri chiziq tekislikni ikkita yarim tekislikka ajratadi.

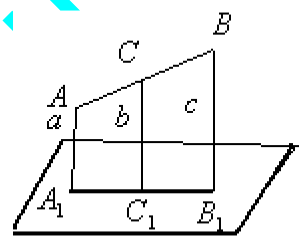
(·) A nuqta**.**



- To'g'ri chiziqda 10 ta kesma (AB, AC, AD, AE, BC, BD, BE, CD, CE, DE) 10 ta nur mavjud (AN, BN, CN, DN, EN, EK, DK, CK, BK, AK).

To'g'ri chiziqdagi n ta nuqtalar tashkil qilgan kesmalar soni $\frac{n(n-1)}{2}$ formula yordamida aniqlanadi.

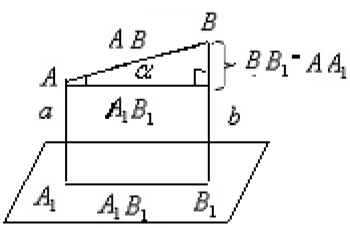
I) Tekislikni kesib o'tmagan hol



$$c = \frac{a+b}{2}$$

A_1B_1-chiziq AB kesmaning tekislikdagi proyeksiyasi

α - AB kesma yotuvchi to'g'ri chiziq va tekislik (yoki AB kesma va tekislik) orasidagi burchak

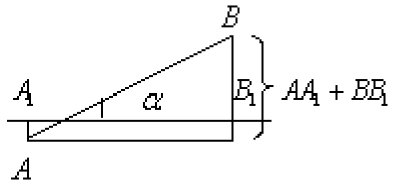


$$A_1B_1^2 + (BB_1 - AA_1)^2 = AB^2; \quad \sin\alpha = \frac{|BB_1 - AA_1|}{AB}; \quad \cos\alpha = \frac{AB_1}{AB};$$

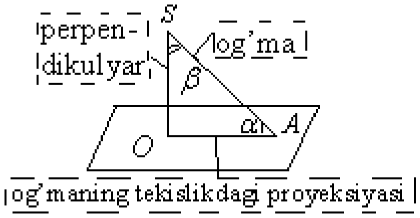
$$\sin\alpha = \frac{|BB_1 - AA_1|}{A_1B_1}$$

II) Tekislikni kesib o'tgan hol

$\sin\alpha = \frac{AA_1 + BB_1}{AB}$; $c = \frac{|AA_1 - BB_1|}{2}$; c-kesma o'rtasidan tekislikkcha masofa



Og'ma, perpendikulyar va tekislik



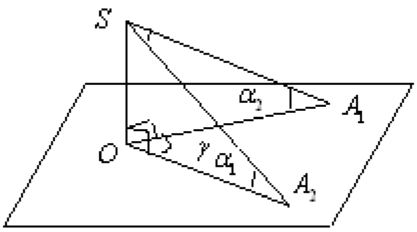
α-og'ma va tekislik orasidagi burchak

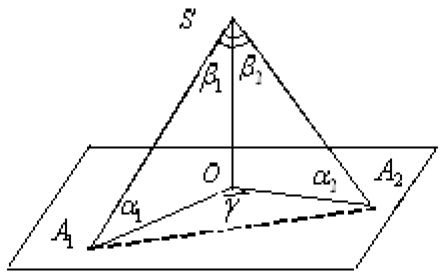
β og'ma va perpendikulyar orasidagi burchak

$SO^2 + OA^2 = SA^2$; $\alpha + \beta = 90^\circ$; $\sin\alpha = \frac{SO}{SA}$; $\cos\alpha = \frac{OA}{SA}$; $tg\alpha = \frac{SO}{OA}$.

PERPENDIKULYAR VA IKKITA OG'MA.

Yon tomondan ko'rinishi:





$SO^2 + OA_1^2 = SA_1^2$; $SO^2 + OA_2^2 = SA_2^2$; $SA_1^2 - A_1^2 = SA_2^2 - OA_2^2$

$A_1A_2^2 = SA_1^2 + SA_2^2 - 2SA_1 \cdot SA_2 \cdot \cos\varphi$

γ-og'malar proyeksiyalari orasidagi burchak

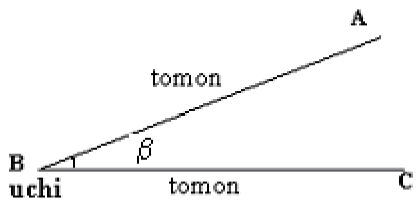
φ-o'gmalar orasidagi burchak;

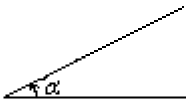
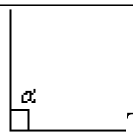
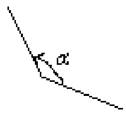
A_1A_2 -og'malar asoslari orasidagi masofa.

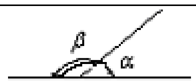
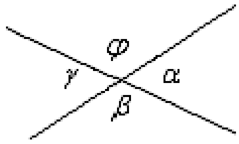
BURCHAKLAR

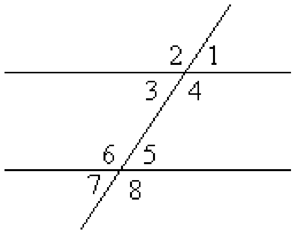
Burchak deb B nuqtadan (burchak uchidan) chiqqan ikki BA va BC nur (burchakning tomonlari) hosil qilgan figuraga aytiladi.

Burchaklar $\angle$A, $\angle$B, $\angle$C,…, yoki $\angle$BAC, $\angle$ABC, $\angle$ACB..., yoki α, β, γ, … orqali belgilandi.

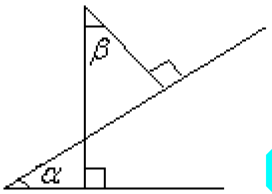
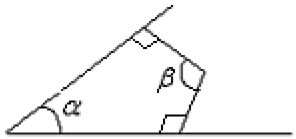
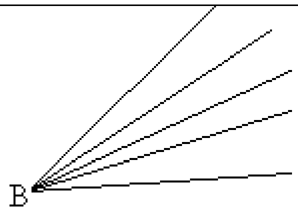


 O'tkir burchak, $0^0 < \alpha < 90^0$.	 To'g'ri burchak, $\alpha = 90^0$.	 O'tmas burchak, $90^0 < \alpha < 180^0$.

 yoyiq burchak, $\alpha=180^0$.	 qo'shni burchaklar, $\beta+\alpha=180^0$.	 α va γ, β va φ vertikal burchaklar, $\alpha=\gamma$, $\beta=\varphi$.



1. Mos burchaklar: 2,6; 1,5; 3,7; 4,8;
2. Ichki almashinuvchi burchaklar: 3,5; 4,6;
3. Tashqi almashinuvchi burchaklar: 1,7; 2,8;
4. Ichki bir tomonli burchaklar: 3,6; 4,5;
5. Tashqi bir tomonli burchaklar: 1,8; 2,7.

 $\alpha=\beta$	 $\alpha+\beta=180^0$	 B nuqtadan chiqqan nurlar soni n ta bo'lsa, burchak soni: $\frac{n(n-1)}{2}$.

Burchaklar radian yoki gradus o'lchovi bilan aniqlanadi.

$1\text{rad}=\frac{180^0}{\pi}=57{,}3^0$; $1^0=\frac{\pi}{180}\text{ rad}=0{,}01745\text{ rad}$;

$a\text{ rad}=\frac{\pi}{180}\cdot\alpha^0$ - gradusdan radian o'lchov birligiga o'tish formulasi;

$\alpha^0=\frac{180^0}{\pi}\cdot a$ - radiandan gradus o'lchov birligiga o'tish formulasi.

KO'PBURCHAK

Agar figuraning ixtiyoriy ikki nuqtasini tutashtiruvchi kesma shu figuraga tegishli nuqtalardan iborat bo'lsa, u qavariq figura deyiladi, aks holda esa botiq figura deyiladi.

Yopiq sodda siniq chiziq tekislikni ikki sohaga ajratadi: ichki va tashqi. Tashqi sohaning ichki sohadan farqi, unda to'la yotadigan biror to'g'ri chiziq doim topiladi. Berilgan siniq chiziq esa shu sohalarning har birining ham chegarasidir. Yopiq sodda siniq chiziq bilan uning ichki sohasi birlashmasi **ko'pburchak** deyiladi. Sodda siniq chiziqning o'zini esa shu ko'pburchakning **chegarasi**, ichki soha nuqtalarini esa ko'pburchakning **ichki nuqtalari** deb aytiladi. Siniq chiziq uchlari ko'pburchakning **uchlari**, uning bo'g'inlari esa ko'pburchak **tomonlari** deb ataladi. Ko'pburchakning ikki qo'shni tomoni orasidagi burchagi uning **ichki burchagi** deyiladi. Ko'pburchakning uchlari, tomonlari va ichki burchaklari soni bir xil bo'lib, shularga asosan uning turlari: uchburchak, to'rtburchak, ..., n-burchak deb aytiladi Ko‘pburchakning barcha tomonlari uzunliklari yig‘indisi uning **perimetri** deb aytiladi. Ko‘pburchakning qo‘shni bo‘lmagan ikki uchini tutashtiruvchi kesma uning **diagonali** deyiladi.

1) Ko‘pburchakning ichki burchaklari yig’indisi: $(n-2)\cdot 180^0$;

2) Ko‘pburchakning tashqi burchaklari yig‘indisi: 360^0;

3) Ko‘pburchakning diagonallari soni: $\frac{n\cdot(n-3)}{2}$, $n \geq 4$;

MUNTAZAM KO‘PBURCHAKLAR.

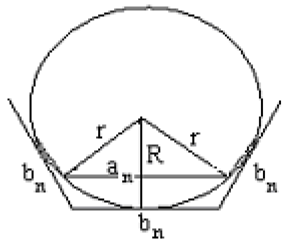
Hamma tomonlari va hamma burchaklari o‘zaro teng qavariq ko‘pburchak **muntazam ko‘pburchak** deyiladi.

Muntazam qavariq ko‘pburchak aylanaga ichki chizilgan bo‘lishi va aylanaga tashqi chizilgan bo‘lishi mumkin.

Rasmda tomoni b_n ga teng bo‘lgan muntazam ko‘pburchakka radiusi r ga teng bo‘lgan aylana ichki chizilgan. Tomoni a_n ga teng bo‘lgan muntazam ko‘pburchakka radiusi R ga teng bo‘lgan aylana tashqi chizilgan.

1) $a_n = 2\cdot R\cdot \sin\frac{180^0}{n}$ - muntazam ko‘pburchakka tashqi chizilgan aylana radiusi R berilgan bo‘lsa, tomonini topish formulasi;

2) $b_n = 2\cdot r\cdot tg\frac{180^0}{n}$ - muntazam ko‘pburchakka ichki chizilgan aylana radiusi r berilgan bo‘lsa, tomonini topish formulasi;



3) $\alpha = \frac{n-2}{n}\cdot 180^0$ - muntazam ko‘pburchakning ichki burchagini hisoblash formulasi;

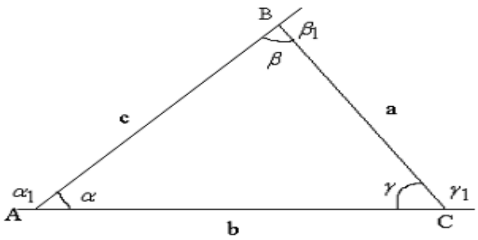
4) $\beta = \frac{360^\circ}{n}$ - muntazam ko‘pburchakning tashqi burchagini hisoblash formulasi;

5) $S_n = p_n\cdot r = \frac{n\cdot b_n\cdot r}{2} = \frac{n\cdot a_n\cdot\sqrt{4R^2 - a_n^2}}{4}$ - muntazam ko‘pburchakning yuzini hisoblash formulasi, bu yerda p_n - yarim perimetr, $p_n = \frac{n\cdot b_n}{2}$.

UCHBURCHAKLAR

Uchburchak deb uch tomonli ko‘pburchakka aytiladi.

α,β,γ-ichki burchaklari, $\alpha_1,\beta_1,\gamma_1$- tashqi burchaklari.



1. $\alpha+\beta+\gamma = 180°$.

2. $\alpha_1+\beta_1+\gamma_1 = 360^\circ$.

3. a+b+c=P, P- perimetr.

BC=a, AC=b,AB=c - uchburchakning tomonlari.

4. $\alpha_1 = \beta+\gamma$, $\beta_1 = \alpha+\gamma$, $\gamma_1 = \alpha+\beta$.

5. Tomonlar orasidagi bog‘lanish: a+b>c; a+c>b; b+c>a.

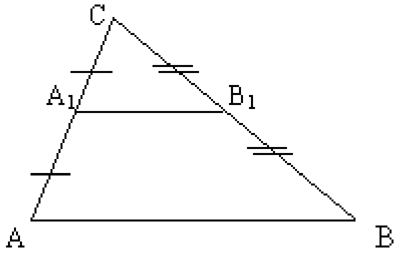
6. O‘tkir burchakli uchburchakda $\alpha+\beta>\gamma$; $\gamma+\beta>\alpha$; $\alpha+\gamma>\beta$ munosabatlar o’rinli.

7. |a-b|<c, |b-c|<a, |a-c|<b.

8. O‘tmas burchakli uchburchakda bitta burchak 90^0 dan katta180^0 dan kichik bo‘lib qolgan 2ta burchagi o‘tkir. $\alpha>\beta+\gamma$, bo‘lsa, mos ravishda α-o‘tmas burchak bo‘lib, uchburchak o‘tmas burchakli uchburchak bo‘ladi.

9. $\alpha=90^0$ bo‘lsa, ucburchak to‘g‘ri burchakli uchburchak deyiladi.

10. ABC uchburchak uchun cosA +cosB+cosC $\leq\frac{3}{2}$ tengsizlik o‘rinli.



O‘RTA CHIZIQ

Uchburchakning ikki tomoni o‘rtalarini tutashtiruvchi kesma uning o‘rta chizig‘i deyiladi.

A_1B_1 - o‘rta chiziq

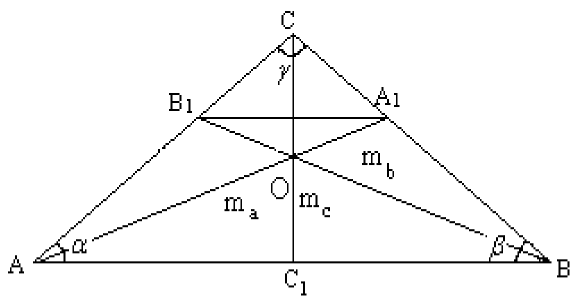
Agar $AA_1=A_1C$ va $BB_1=B_1C$ bo‘lsa, $A_1B_1 \| AB$ va $A_1B_1=AB/2$ bo‘ladi.

MEDIANA

Uchburchakning uchini uning qarchisidagi tomon o‘rtasi bilan tutashtiruvchi kesmaga uchburchakning **medianasi** deyiladi.

$AC_1=C_1B$ bo‘lsa, CC_1-mediana;

$AB_1=B_1C$ bo‘lsa, BB_1-mediana;



$BA_1=A_1C$ bo‘lsa, AA_1-mediana.

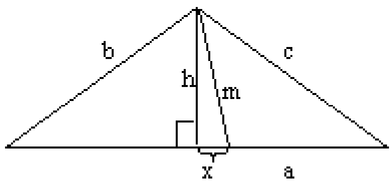
$AO:OA_1=BO:OB_1=CO:OC_1=2:1$.

$\angle A=\alpha$, $\angle B=\beta$, $\angle C=\gamma$;

1) $AA_1=m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}=\frac{1}{2}\sqrt{b^2+c^2+2bc\cos\alpha}$;

$BB_1=m_b=\frac{1}{2}\sqrt{2a^2+2c^2-b^2}=\frac{1}{2}\sqrt{a^2+c^2+2ac\cos\beta}$;

$CC_1=m_c=\frac{1}{2}\sqrt{2b^2+2a^2-c^2}=\frac{1}{2}\sqrt{a^2+b^2+2ab\cos\gamma}$;



2) $m^2_a+m^2_b+m^2_c=\frac{3}{4}(a^2+b^2+c^2)$;

3) $b=\frac{2}{3}\sqrt{2(m_a^2+m_c^2)-m_b^2}$; $a=\frac{2}{3}\sqrt{2(m_b^2+m_c^2)-m_a^2}$; $c=\frac{2}{3}\sqrt{2(m_a^2+m_b^2)-m_c^2}$;

4) $S=\frac{1}{3}\sqrt{(m_a+m_b+m_c)(m_b+m_c-m_a)(m_a+m_c-m_b)(m_a+m_b-m_c)}$

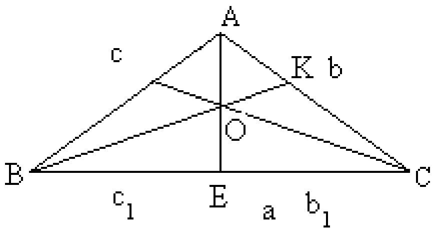
5) ΔABC da $A(x_1;y_1;z_1)$, $B(x_2;y_2;z_2)$, $C(x_3;y_3;z_3)$ bo‘lsa, medianalar kesishish nuqtasi $M(x_0,y_0,z_0)$ quyidagicha aniqlanadi: $x_0=\frac{x_1+x_2+x_3}{3}$, $y_0=\frac{y_1+y_2+y_3}{3}$, $z_0=\frac{z_1+z_2+z_3}{3}$.

6) Balandlik va mediana ajratgan kesma (ixtiyoriy uchburchak uchun): $x=\frac{|b^2-c^2|}{2a}$ (a-asos).

$h^2=b^2-\left(\frac{a}{2}-x\right)^2$, $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ va $x^2=m^2-h^2$ tengliklardan yuqoridagi formula kelib chiqadi.

BISSEKTRISA

Uchburchakning berilgan uchidan o’tkazilgan **bissektrisasi** deb uchburchak burchagi bissektrisasining shu uchni uning qarshi tomondagi nuqta bilan tutashtiruvchi kesmasiga aytiladi.



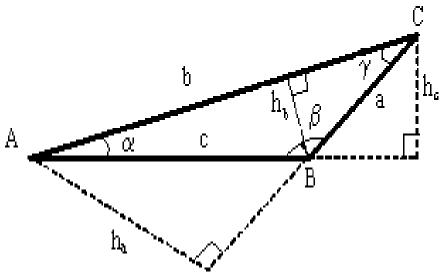
$\angle BAE=\angle EAC$ bo‘lsa, $AE=l_a$ bissektrisa bo‘ladi.

Bissektrisa xossalari:

BE:EC= AB:AC yoki $c_1:b_1=c:b$.

$\frac{OB}{OK}=\frac{AB}{AK}$, $\frac{AO}{OE}=\frac{b+c}{a}$;

$\angle A=\alpha$, $\angle B=\beta$, $\angle C=\gamma$;

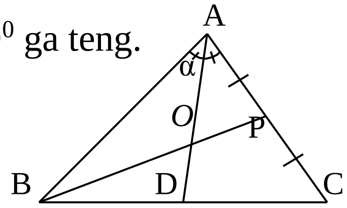


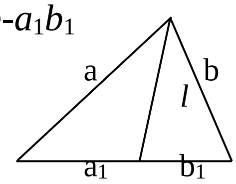
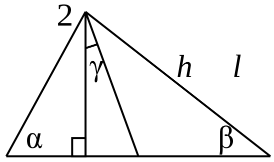
$$l_a=\frac{2\sqrt{bcp(p-a)}}{b+c}=\frac{\sqrt{2b^2c^2(1+\cos\alpha)}}{b+c}=\frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c}$$

$$=\frac{2bc\cos\frac{\alpha}{2}}{b+c}=\frac{4R\sin\beta\cos\frac{\alpha}{2}}{\sin\beta+\sin\gamma},$$

bu yerda R-uchburchakka tashqi chizilgan aylananing radiusi, $p=(a+b+c)/2$ – yarim perimetr.

Qo‘shni burchaklar bissektrisasi orasidagi burchak 90^0 ga teng.



$l^2=ab-a_1b_1$  	$\gamma=\frac{\lvert\alpha-\beta\rvert}{2}$  	AC=BC; AP=PC; $\frac{OA}{OD}=\frac{OB}{OP}$

BALANDLIK

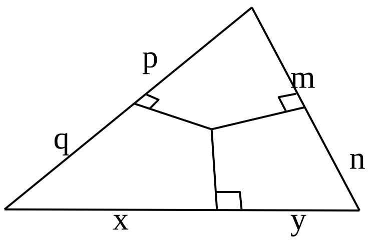
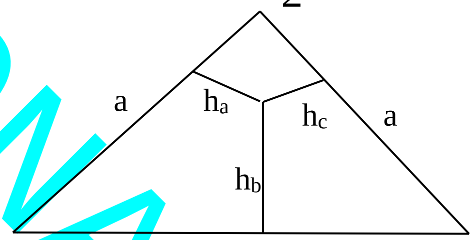
Uchburhcak berilgan uchidan tushirilgan **balandligi** deb uchburchakning shu uchidan uning qarshisidagi tomoni yotgan to‘g‘ri chiziqqa tushirilgan perpendikulyarga aytiladi.

1) $h_a=\frac{2S}{a}\sqrt{(2ac)^2-(a^2+c^2-b^2)}=b\cdot\sin\gamma=c\cdot\sin\beta;$

$h_b=\frac{2S}{b}\sqrt{(2bc)^2-(b^2+c^2-a^2)}=a\cdot\sin\beta=b\cdot\sin\alpha;$

2) $h_a:h_b:h_c=\frac{1}{a}:\frac{1}{b}:\frac{1}{c}=bc:ac:ab;$

3) $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},$ r – ichki chizilgan aylana radiusi.

$x^2+n^2+p^2=y^2+q^2+m^2$  	$h_a+h_b+h_c=\frac{a\sqrt{3}}{2}$ 

UCHBURCHAKKA ICHKI VA TASHQI CHIZILGAN AYLANA

1) Uchburchakk ichki chizilgan aylana markazi bissektrisalar kesishgan nuqtada bo‘ladi.

2) Uchburchakka tashqi chizilgan aylananing markazi o‘rta perpendikulyarlar kesishgan nuqtada bo‘ladi.

3) $d^2=R^2-2Rr$, d – uchburchakka ichki va tashqi chizilgan aylanalar markazlari orasidagi masofa.

4) $r=\frac{S}{p}=\frac{\sqrt{p(p-a)(p-b)(p-c)}}{p};$ $p=\frac{a+b+c}{2};$ r – uchburchakka ichki chizilgan aylana radiusi.

$$r=(p-a)tg\frac{\alpha}{2}=(p-b)tg\frac{\beta}{2}=(p-c)tg\frac{\gamma}{2}=p\cdot tg\frac{\alpha}{2}\cdot tg\frac{\beta}{2}\cdot tg\frac{\gamma}{2}=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

Agar uchburchak teng yonli bo‘lsa, $r=\frac{a}{2}tg\frac{\alpha}{2};$ a – asos; $2\alpha+\beta=180^0$.

5) $R=\dfrac{abc}{4S}=\dfrac{abc}{4\sqrt{p(p-a)(p-b)(p-c)}}$; $p=\dfrac{a+b+c}{2}$, R – tashqi chizilgan aylana radiusi. $R=\dfrac{p}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$; $R=\dfrac{a}{2\sin\alpha}$. Bunda α – a tomon qarshisidagi burchak.

UCHBURCHAK YUZI

1) $S=\dfrac{a\cdot h_a}{2}=\dfrac{b\cdot h_b}{2}=\dfrac{c\cdot h_c}{2}$ – tomon va unga tushirilgan balandlik orqali.

2) $S=\dfrac{abc}{4R}=p\cdot r;\ p=\dfrac{a+b+c}{2}$; – ichki (r) va tashqi (R) chizilgan aylana radiuslari orqali.

3) $S=\dfrac{4}{3}\sqrt{M(M-m_a)(M-m_b)(M-m_c)}$, bu yerda $M=\dfrac{m_a+m_b+m_c}{2}$.

$$S=\frac{1}{3}\sqrt{(m_a+m_b+m_c)(m_b+m_c-m_a)(m_a+m_c-m_b)(m_a+m_b-m_c)}$$

m_a, m_b, m_c - medianalar.

4) $S=\sqrt{p(p-a)(p-b)(p-c)}$; $p=\dfrac{a+b+c}{2}$ – *Geron formulasi.*

5) $S=r\left(\dfrac{r}{tg\frac{\alpha}{2}}+2R\sin\alpha\right)$, $S=2R^2\sin\alpha\sin\beta\sin\gamma$.

5) $S=\dfrac{1}{\sqrt{\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\left(\frac{1}{h_b}+\frac{1}{h_c}-\frac{1}{h_a}\right)\left(\frac{1}{h_a}+\frac{1}{h_c}-\frac{1}{h_b}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}-\frac{1}{h_c}\right)}}$; h_a, h_b, h_c balandliklar.

6) $S=\dfrac{1}{2}ab\sin\alpha=\dfrac{1}{2}ac\sin\beta=\dfrac{1}{2}bc\sin\gamma$ – ikki tomon va ular orasidagi burchak orqali.

7) $S=\dfrac{a^2\sin\beta\sin\gamma}{2\sin\alpha}=\dfrac{b^2\sin\alpha\sin\gamma}{2\sin\beta}=\dfrac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$.

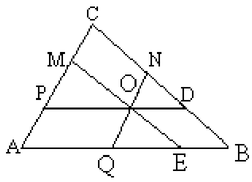
8) ΔABC uchlari $A(x_1,y_1), B(x_2,y_2)$ ââ $C(x_3,y_3)$ nuqtalarda bo'lsa, uchburchakning yuzi quyidagicha aniqlanadi:

$$S=\frac{1}{2}\left(|x_1y_2-x_2y_1|+|x_2y_3-x_3y_2|+|x_3y_1-x_1y_3|\right)$$

9) $S_{\Delta NOD}=S_1,\ S_{\Delta OQE}=S_2,\ S_{\Delta POM}=S_3$

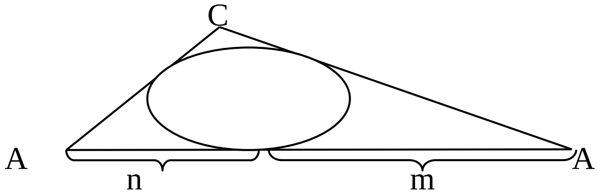
$AB \parallel PD;\ AC \parallel QN;\ CB \parallel ME$ bo'lsa,

$S_{\Delta ABC}=\left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_2}\right)^2.$



10) $\angle C=90°$

$S=mn.$



UCHBURCHAKLAR UCHUN ASOSIY TEOREMALAR

1) Sinuslar teoremasi. (Bunda R – tashqi chizilgan aylana radiusi.)

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=2R,\quad a=2R\sin\alpha,\ b=2R\sin\beta,\ c=2R\sin\gamma.$$

2) Kosinuslar teoremasi.

$$a^2=b^2+c^2-2bc\cos\alpha,\quad b^2=a^2+c^2-2ac\cos\beta,\ c^2=a^2+b^2-2ab\cos\gamma,$$

$$a=b\cos\gamma+c\cos\beta,\qquad b=a\cos\gamma+c\cos\alpha,\qquad c=a\cos\beta+b\cos\alpha.$$

3) Tangenslar teoremasi.

$$\frac{a+b}{a-b}=\frac{tg\frac{a+b}{2}}{tg\frac{a-b}{2}}=\frac{ctg\frac{\gamma}{2}}{tg\frac{a-b}{2}}.$$

4) Molveyde formulasi

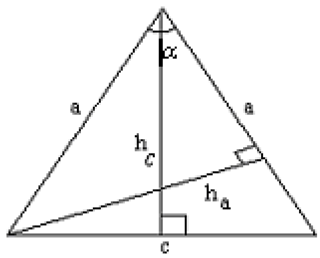
$$\frac{a+b}{2}=\frac{\cos\frac{a-b}{2}}{\sin\frac{\gamma}{2}}.$$

5) $\sin\frac{\alpha}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}},\qquad \cos\frac{\alpha}{2}=\sqrt{\frac{p(p-a)}{bc}}.$

XUSUSIY HOLLAR

TENG YONLI UCHBURCHAK

Agar uchburchakning ikki tomoni teng bo'lib, uchinchi tomoni teng bo'lmasa, u **teng yonli uchburchak** deyiladi. Bu teng tomonlar uchburchakning yon tomonlari, uchinchi tomoni esa uchburchakning asosi deyiladi.



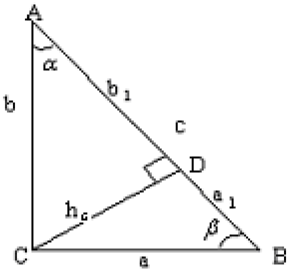
$$h_a=\frac{\sqrt{4a^2-c^2}}{2},\qquad h_a=c\cos\frac{\alpha}{2}=2h_c\sin\frac{\alpha}{2},$$

$$R=\frac{a^2}{2h_c},\qquad r=\frac{c(2a-c)}{4h_c}.$$

TO‘G‘RI BURCHAKLI UCHBURCHAK

Agar uchburchakning to‘g‘ri burchagi bo‘lsa, u **to‘g‘ri burchakli uchburchak** deyiladi. To‘g‘ri burchakli uchburchakning to‘g‘ri burchagi qarshisidagi tomoni gipotenuza, qolgan ikki tomoni katetlar deb ataladi.

To‘g‘ri burchakli uchburhcakda 30^0 li burchak qarshisida yotgan katet gipotenuzaning yarmiga teng.



$$h_c^2 = a_1 \cdot b_1 = AD \cdot DB,$$

$$a^2 = a_1 \cdot c, \quad b^2 = b_1 \cdot c,$$

$$c = a_1 + b_1, \quad \frac{a_1}{b_1} = \frac{a^2}{b^2},$$

$$\sin\alpha = \frac{a}{c} = \cos\beta, \quad \cos\alpha = \frac{b}{c} = \sin\beta,$$

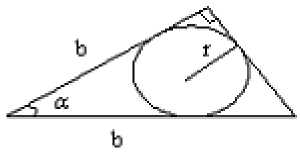
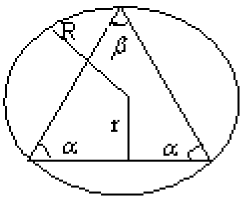
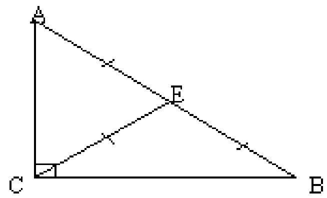
$tg\alpha = \dfrac{a}{b} = ctg\beta$. Agar $\dfrac{R}{r} = \dfrac{5}{2}$ bo'lsa, $a:b:c = 3:4:5$ bo'ladi.

$$S = \frac{ab}{2} = \frac{ch_c}{2} = \frac{a^2 \cdot ctg\alpha}{2} = \frac{c^2 \cdot \sin 2\alpha}{4} = r^2 + 2rR.$$

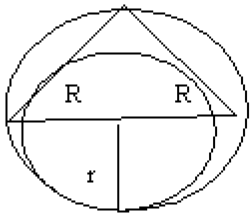
Agar $\sin\alpha + \sin\beta = q$ bo’lsa, $S = \dfrac{1}{4}c^2\left(q^2 - 1\right)$ o’rinli.

To'g'ri burchakli uchburchakka ichki chizilgan kvadrat perimetri:

$$P = \frac{4ab}{a+b}; \; a, b - \text{katet}.$$

$S = \dfrac{c\sqrt{4a^2 - c^2}}{4}$.	$r = b\sin\dfrac{\alpha}{2} tg\left(\dfrac{\pi}{4} + \dfrac{\alpha}{4}\right)$. 
$S = 4R^2 \sin^2\alpha \cdot \cos\dfrac{\beta}{2} \cdot \sin\dfrac{\beta}{2}$; $\dfrac{r}{R} = \sin 2\alpha \cdot tg\dfrac{\alpha}{2}$.  	R=m$_c = \dfrac{c}{2}$ CE-mediana  $r = \dfrac{a+b-c}{2}, \; R = \dfrac{c}{2}$ $r + R = \dfrac{a+b}{2}$

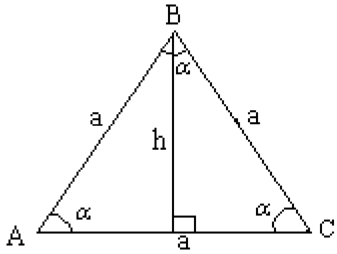
Agar $2R+r=\frac{a+b+c}{2}$ bajarilsa, uchburchak to‘g‘ri burchakli uchburchak bo‘ladi.



Pifagor teoremasi: $c^2=a^2+b^2$, $c^2-a^2=b^2$, $c^2-b^2=a^2$.
To‘g‘ri burchakli uchburchakka tashqi chizilgan aylananing diametri $(2R=D)$ gipotenuza (c) ga teng.

$r=2R\left(\sqrt{2}-1\right)$

TENG TOMONMLI (MUNTAZAM) UCHBURCHAK.



Hamma tomonlari teng uchburchak teng tomonli uchburchak deyiladi.

$AB=BC=AC=b=c=a,$

$\angle A=\angle B=\angle C=\alpha=60^0$, $P=3a-$perimetri.

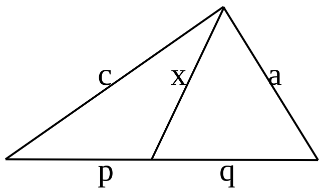
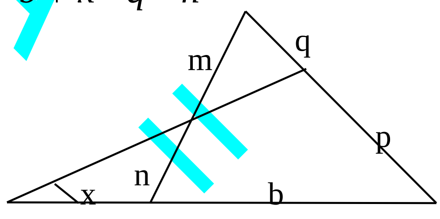
$h_a=h_b=h_c=l_a=l_b=l_c=m_a=m_b=m_c=3r=1{,}5R=\left(\frac{\sqrt{3}}{2}\right)a=r+R.$

h_a - balandlik, l_a - bissektrisa, m_a - mediana, r - ichki chizilgan aylana radiusi, R - tashqi chizilgan aylana radiusi.

$R=\frac{\sqrt{3}a}{3}$, $r=\frac{\sqrt{3}a}{6}=\frac{R}{2}$, $R=2r$, $S=\frac{\sqrt{3}}{4}a^2$, $a=\sqrt{3}R=2\sqrt{3}r.$

h_1, h_2, h_3 - ΔABC ning ichidagi ixtiyoriy nuqtasidan tomonlarigacha bo‘lgan balandliklari, h - muntazam ΔABC ning balandligi bo‘lsa, $h_1+h_2+h_3=h$ tenglik o‘rinli bo‘ladi.

IXTIYORIY UCHBURCHAK

$x^2=\frac{a^2p+c^2q}{p+q}-pq$ 	uchburchakning medianalari va bissektrisalari kesishgan nuqtalar orasidagi masofa $d=\sqrt{\left(r-\frac{1}{3}h_c\right)^2+\frac{1}{9}(a-b)^2}$	$\frac{x}{b+x}\cdot\frac{p}{q}\cdot\frac{m}{n}=1$ 

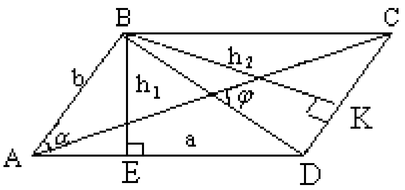
TO‘RTBURCHAKLAR

To‘rtta nuqta va bu nuqtalarni ketma-ket tutashtiruvchi to‘rtta kesmadan iborat figura **to‘rtburchak** deyiladi. Bunda nuqtalardan hech qanday uchtasi bir to‘g‘ri chiziqda yotmasligi, ularni tutashtiruvchi kesmalar esa kesishmasligi kerak. Berilgan nuqtalar to‘rtburchakning uchlari, ularni tutashtiruvchi kesmalar esa tomonlari deyiladi.

PARALLELOGRAMM

Qarama-qarshi tomonlari parallel bo‘lgan to‘rtburchak **parallelogramm** deyiladi.

Parallelogrammning diagonallari kesishadi va kesishish nuqtasida teng ikkiga bo‘linadi.



Parallelogrammning qarama-qarshi tomonlari, qarama-qarshi burchaklari teng.

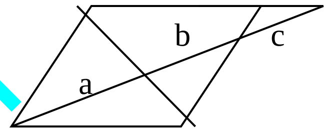
1) $AB \parallel CD,\ AB = CD,\quad BC \parallel AD,\quad \angle A = \angle C,\ \angle B = \angle D,\quad \angle A + \angle B = 180^0.$

2) $AC^2 + BD^2 = 2\left(AB^2 + BC^2\right)$ yoki $d_1^2 + d_2^2 = 2a^2 + 2b^2.$

3) $BE \perp AD,\ BE = h_1$ - balandlik, $BK \perp CD,\ BK = h_2$ - balandlik.

$$S = ah_1 = bh_2 = a \cdot b \cdot \sin\alpha = \frac{d_1 \cdot d_2 \cdot \sin\varphi}{2}.$$

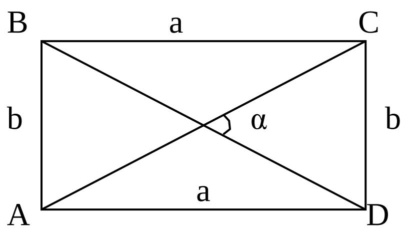
$a^2 = b(b + c)$.



TO‘G‘RI TO‘RTBURCHAK.

To‘g‘ri to‘rtburchak hamma burchaklari to‘g‘ri bo‘lgan parallelogrammdir.

To‘g‘ri to‘rtburchakning diagonallari teng.



1) $AB = CD = b,\quad BC = AD = a,\quad \angle A = \angle B = \angle C = \angle D = 90^0.$

2) $d = AC = BD$ - diagonal, $d = AC = BD = \sqrt{a^2 + b^2} = 2R.$

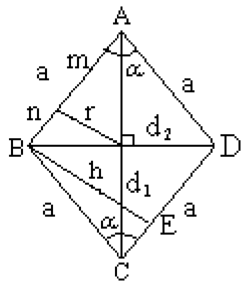
3) $R = \dfrac{d}{2} = \dfrac{1}{2}\sqrt{a^2 + b^2}.$

4) $S = a \cdot b = \dfrac{1}{2} d^2 \sin\alpha = 2R^2 \sin\alpha.$

ROMB

Romb hamma tomonlari teng bo‘lgan parallelogrammdir.

Rombning diagonallari to‘g‘ri burchak ostida kesishadi. Romb diagonallari uning burchaklari bissektrisalaridir.



1) $AB \| CD,\ BC \| AD,\quad AB = BC = CD = AD = a = m + n$.

2) $AC \perp BD,\ BE \perp CD,\ BE = 2r = h.$

3) $\angle BAD = \angle BCD,\quad \angle ABC = \angle ADC,\quad \angle BAD + \angle ABC = 180^0.$

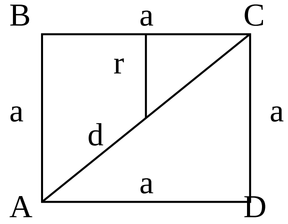
4) $d_1^2 + d_2^2 = 4a^2;\ d_1 = 2a\cos\dfrac{\beta}{2};\ d_2 = 2a\sin\dfrac{\beta}{2};\ a = m + n$ bo‘lsa,

$r^2 = m \cdot n.$

5) $S = a \cdot h = a^2 \cdot \sin\alpha = \dfrac{1}{2} \cdot d_1 \cdot d_2 = 2a \cdot r,\quad S = \dfrac{(d_1 + d_2)^2 - \left(\dfrac{P}{2}\right)^2}{4} = \dfrac{4a^2 + 2d_1d_2 - \left(\dfrac{P}{2}\right)^2}{4},$

$S = \dfrac{8r^2}{\sqrt{3}}.$

KVADRAT.



Kvadrat hamma tomonlari teng bo‘lgan to‘g‘ri to‘rtburchakdir.

Kvadratning barcha burchaklari to‘g‘ri burchaklar.

Kvadratning diagonallari teng.

Kvadratning diagonallari to‘g‘ri burchak ostida kesishadi va uning burchaklari bissektrisalari bo‘ladi.

$AB = BC = CD = AD = a, \qquad \angle A = \angle B = \angle C = \angle D = 90^0.$

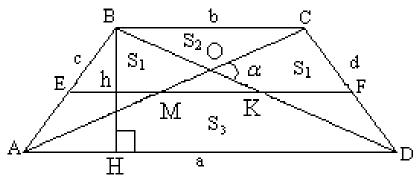
$d = AC = BD$ - diagonal, $d = 2R = \sqrt{2}a = 2\sqrt{2}r.$

$r = \frac{a}{2} = \frac{R}{\sqrt{2}} = \frac{d}{2\sqrt{2}}, \quad R = \frac{\sqrt{2}}{2}a = \frac{d}{2} = \sqrt{2}r, \qquad S = a^2 = \frac{1}{2}d^2 = 4r^2 = 2R^2,$

bu yerda R - kvadratga tashqi chizilgan aylana radiusi, - r ichki chizilgan aylana radiusi.

TRAPETSIYA

Ikkita qarama-qarshi tomolarigina prallel bo‘lgan to‘rtburchak **trapetsiya** deb ataladi. Bu parallel tomonlar trapetsiyaning **asoslari** deyiladi. Boshqa ikki tomon esa uning **yon tomonlari** deyiladi.



Yon tomonlari teng trapetsiya **teng yonli trapetsiya** deyiladi.

Trapetsiya yon tomonlarining o‘rtalarini tutashtiruvchi kesma trapetsiyaning **o‘rta chizig‘i** deyiladi.

Trapetsiyaning o‘rta chizig‘i asoslariga parallel va ular yig‘indisining yarmiga teng.

1) $AD \parallel BC$, $BH = h$, $AC = d_1$, $BD = d_2$, $AD = a$, $BC = b.$

2) $EF = \frac{a+b}{2}$ - o‘rta chiziq, $AE = EB$, $CF = FD$, $EF \parallel AD$, $EM = KF = \frac{b}{2}$,

$EK = MF = \frac{a}{2}, \qquad MK = \frac{a-b}{2}.$

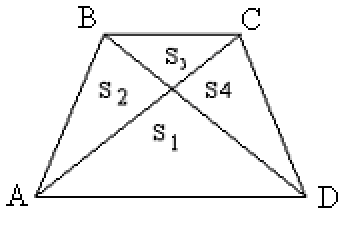
3) Agar trapetsiyaga ichki aylana chizish mumkin bo‘lsa, $S = p \cdot r$ tenglik o‘rinli bo‘ladi, bu yerda $p = \frac{a+b+c+d}{2}$.

4) $S = \frac{a+b}{2} \cdot h = \frac{1}{2} d_1 \cdot d_2 \cdot \sin\alpha = EF \cdot h$

5) Trapetsiya teng yonli bo‘lsa, $AB = CD = c$, $AH = \frac{a-b}{2}$, $HD = \frac{a+b}{2}$ bo‘ladi.

$\frac{AO}{OC} = \frac{DO}{OB} = \frac{a}{b}.$

a) $d = \sqrt{ab + c^2}$, b) $AC \perp BD$ bo‘lsa, $d = \frac{\sqrt{2}}{2}(a+b)$,



c) $S=\dfrac{(a+b)^2}{4}$, $\left(\alpha=90^0\right)$.

6) $S_{\Delta ABO}=S_{\Delta COD}=S_1$, $S_{\Delta BOC}=S_2$, $S_{\Delta AOD}=S_3$, $S_1^2=S_2\cdot S_3$.

7) $a+b=c+d$ tenglik bajarilsa, trapetsiyaga ichki aylana chizish mumkin. $c=d$ tenglik bajarilsa, unga tashqi aylana chizish mumkin.

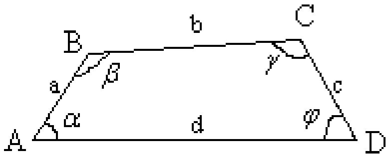
$2R=H=\sqrt{ab}$

$S_2=S_4=\sqrt{S_1\cdot S_3}$

$S=\left(\sqrt{S_1}+\sqrt{S_3}\right)^2$

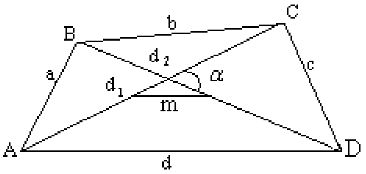
AC⊥BD $d_1=d_2=d$ bo‘lsa, $\dfrac{d^2}{2}=h^2$.

IXTIYORIY TO`RTBURCHAK



1) $\angle A+\angle B+\angle C+\angle D=\alpha+\beta+\gamma+\varphi=360^0$.

$p=a+b+c+d$ - perimetri.



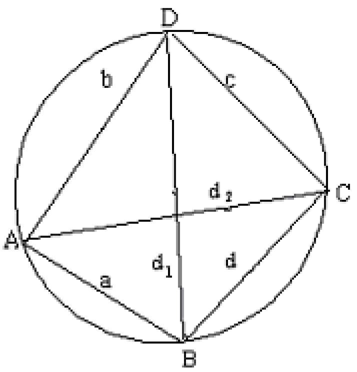
m - AC va BD diagonallari o‘rtasini tutashtiruvchi kesma. $AC=d_1$, $BD=d_2$ bo‘lsa,

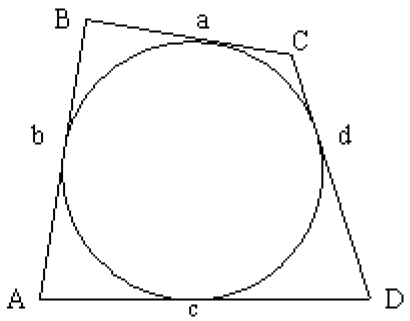
$a^2+b^2+c^2+d^2=d_1^2+d_2^2+4m^2$. $S=\dfrac{1}{2}d_1\cdot d_2\cdot\sin\alpha$.

Agar $\angle BAD+\angle BCD=\angle ABC+\angle ADC=180^0$ bo‘lsa, ABCD to‘rt burchakka tashqi aylana chizish mumkin va $a\cdot c+b\cdot d=d_1\cdot d_2$

$S=\sqrt{(p-a)(p-b)(p-c)(p-d)}$, $p=\dfrac{a+b+c+d}{2}$, $R=\dfrac{1}{4S}\sqrt{(ad+cd)(ac+bd)(ad+bc)}$.

Agar ABCD to‘rtburchakka ichki va tashqi aylana chizish mumkin bo‘sa $S=\sqrt{a\cdot b\cdot c\cdot d}$ o‘rinli bo‘ladi.





Agar $a+c=b+d$ bajarilsa, ABCD to‘rtburchakka ichki aylana chizish mumkin va $S=p\cdot r$, $p=\dfrac{a+b+c+d}{2}$, $S=\sqrt{p(p-a)(p-b)(p-c)(p-d)}=(a+c)r=(b+d)r$.

MUNTAZAM BESHBURCHAK.

Ichki burchagi - 108^0.

Ichki burchaklari yig'indisi – 540^0 .

Tashqi burchagi - 72^0.

$$n=5,\quad a=R\sqrt{\frac{5-\sqrt5}{2}}=\frac{R}{2}\sqrt{10-2\sqrt5}=2r\sqrt{5-2\sqrt5},\quad R=\frac{a}{10}\sqrt{50+10\sqrt5}=r\left(\sqrt5-1\right),$$

$$r=\frac{a}{10}\sqrt{25+10\sqrt5}=\frac{R}{4}\left(\sqrt5+1\right),\quad d=\frac{1+\sqrt5}{2},$$ bunda d - diagonal.

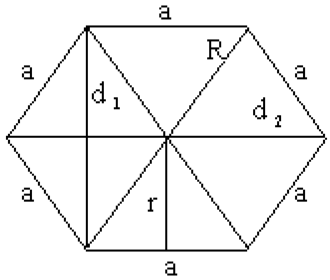
$$S=\frac{5}{8}R^2\sqrt{10+2\sqrt5}=\frac{a^2}{4}\sqrt{25+10\sqrt5}=5r^2\sqrt{5-2\sqrt5}.$$

MUNTAZAM OLTIBURCHAK.

Ichki burchagi - 120^0.

Ichki burchaklari yig'indisi – 720^0 .

Tashqi burchagi - 60^0



$$n=6,\quad a=R=\frac{2\sqrt3}{3}r,$$

$$r=\frac{\sqrt3}{2}R=\frac{\sqrt3}{2}a=\frac{R}{4}\left(\sqrt5+1\right)=\frac{a}{10}\sqrt{25+10\sqrt5},$$

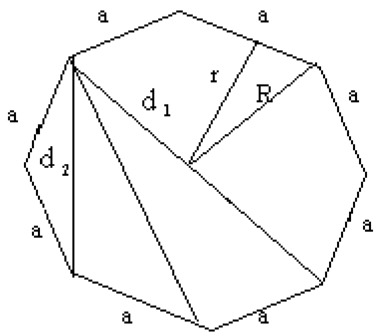
$$d_1=\sqrt3 a,\qquad d_2=2a=2R,\quad S=\frac{3}{2}R^2\sqrt3=\frac{3}{2}a^2\sqrt3=2r^2\sqrt3.$$

MUNTAZAM SAKKIZBURCHAK.

Ichki burchagi - 135^0.

Ichki burchaklari yig'indisi – 1080^0 .

Tashqi burchagi - 45^0



$$a=R\sqrt{2-\sqrt2}=2r\left(\sqrt2-1\right),\quad R=\frac{a}{R}\sqrt{4+2\sqrt2}=r\sqrt{4-2\sqrt2},$$

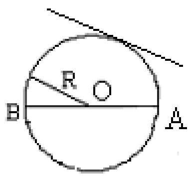
$$r=\frac{R}{2}\sqrt{2+\sqrt2}=\frac{a}{2}\left(\sqrt2+1\right),$$

$$d_1=a\sqrt{2+\sqrt2},\ d_2=a(1+\sqrt2),\quad d_1=2R=\frac{a}{\sqrt{2-\sqrt2}},$$

$$S=2R^2\sqrt2=2a^2\left(1+\sqrt2\right)=8r^2\left(\sqrt2-1\right).$$

AYLANA

Tekislikning berilgan nuqtadan bir xil uzoqlashgan hamma nuqtalaridan iborat figura aylana deyiladi. Berilgan nuqta aylananing markazi deyiladi.



Aylana nuqtalaridan uning markazigacha masofa aylananing radiusi deyiladi. Aylana nuqtasini uning markazi bilan tutashtiruvchi har qanday kesma ham radius deyiladi.

Aylananing ikkita nuqtasini tutashtiruvchi kesma vatar deyiladi. Aylana markazidan o'tuvchi vatar diametr deyiladi.

Aylananing nuqtasidan uning shu nuqtaga o'tkazilgan radiusiga perpendikulyar holda o'tuvchi to'g'ri chiziq aylanaga urinma deyiladi. Bunda aylananing bu nuqtasi urinish nuqtasi deyiladi.

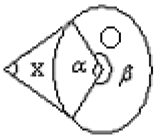
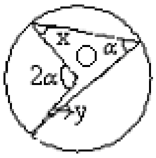
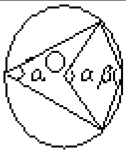
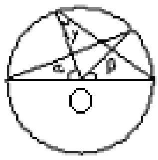
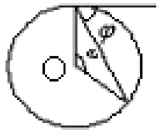
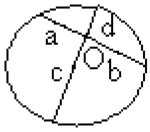
Aylana uzunligi - $C = 2\pi R = \pi D$, $D = 2R$, D - diametr.

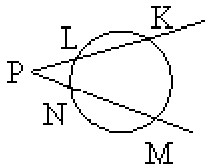
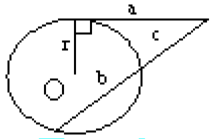
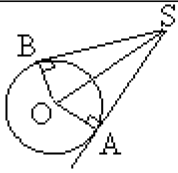
Markazi koordinata boshida, radiusi R bo‘lgan aylana tenglamasi: $x^2 + y^2 = R^2$.

Markazi $A(a;b)$ nuqtada, radiusi R bo‘lgan aylana tenglamasi: $(x-a)^2 + (y-b)^2 = R^2$.

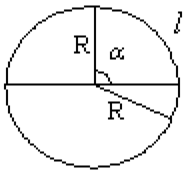
Markazi A$(a;b;c)$ nuqtada, radiusi R bo‘lgan sfera tenglamasi: $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$.

AYLANA VA DOIRADAGI BURCHAKLAR

 $x = \frac{\beta - \alpha}{2}$	 $\alpha = x + y$	 $\beta = 180^0 - \frac{\alpha}{2}$	 $\gamma = \frac{\alpha + \beta}{2}$	 $\varphi = \frac{\alpha}{2}$	 $a \cdot b = c \cdot d$

 $PL \cdot PK = PN \cdot PM$	 $a^2 = (b + c) \cdot c$	 $AS = BS$, $\angle ASO = \angle BSO$

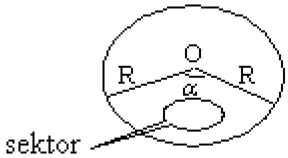
Doiraviy figuralar

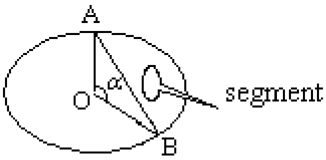


$l = \frac{\alpha \pi R}{180^0} = a_{rad} \cdot R$, l - yoy uzunligini hisoblash formulasi.

$S = \pi \cdot R^2 = \frac{1}{4}\pi \cdot d^2$ - doira yuzini hisoblash formulasi, $2R = d$.

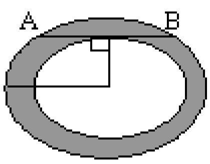
$S = \frac{\pi R^2 \cdot \alpha}{360^0} = \frac{R^2 a_{rad}}{2}$ - doiraviy sektor yuzini hisoblash formulasi.





$S = \frac{R^2}{2}\left(\frac{\pi\alpha}{180^0} - \sin\alpha\right)$ - segment yuzini hisoblash formulasi.

$S_{halqa} = \pi\left(R^2 - r^2\right) = \pi\left(\frac{AB}{2}\right)^2$ - halqa yuzini hisoblash formulasi.



Stereometriya

Stereometriya – geometriyaning bir bo‘limi bo‘lib, unda fazodagi figuralar o‘rganiladi.

Stereometriya aksiomalari:

- Tekislik qanday bo‘lmasin shu tekislikka tegishli nuqtalar va unga tegishli bo‘lmagan nuqtalar mavjud.

- Agar ikkita tekislik umumiy nuqtaga ega bo‘lsa, ular shu nuqtadan o‘tuvchi to‘g‘ri chiziq bo‘yicha kesihsadi.

- Agar ikkita turli to‘g‘ri chiziq umumiy nuqtaga ega bo‘lsa, ular orqali bitta va faqat bitta tekislik o‘tkazish mumkin.

KO’PYOQ

Sirti chekli miqdordagi yassi tekisliklardan iborat jism **ko‘pyoq** deyiladi. Agar ko‘pyoqning o‘zi uning sirtidagi har bir ko‘pburchak tekisligining bir tomonida yotsa, bunday ko‘pyoq **qavariq ko‘pyoq** deyiladi. Qavariq ko‘pyoqning yoqlari yassi qavariq ko‘pburchaklardan iborat. Ko‘pyoq yoqlarining tomonlari uning qirralari, uchlari esa ko‘pyoqning uchlari deyiladi.

PRIZMA

Turli tekisliklarda yotuvchi va parallel ko‘chirish bilan ustma-ust tushuvchi ikkita yassi ko‘pburchakdan hamda bu ko‘pburchaklarning mos nuqtalarini tutashtiruvchi hamma kesmalardan iborat ko‘pyoq **prizma** deyiladi. Ko‘pburchaklar **prizmaning asoslari** deyiladi, mos uchlarni tutashtiruvchi kesmalar esa **prizmaning yon qirralari** deyiladi. Prizmaning yon yoqlari parallelogramm.

Prizmaning sirti asoslaridan va yon sirtidan iborat. Yon sirti parallelogrammlardan iborat.

Prizma asoslarining tekisliklari orasidagi masofa **prizmaning balandligi** deyiladi. Prizmaning bitta yog‘iga tegishli bo‘lmagan ikki uchini tutashtiruvchi kesma **prizmaning diagonali** deyiladi.

Agar prizmaning asosi *n* burchakli bo‘lsa, u ***n* burchakli prizma** deyiladi.

Agar prizmaning yon qirrasi asoslariga perpendikulyar bo‘lsa, bunday prizma **to‘g‘ri prizma** deyiladi, aks holda **og‘ma prizma** deyiladi.

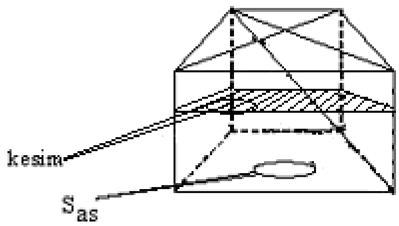
Agar to‘g‘ri prizmaning asoslari muntazam ko‘pburchaklar bo‘lsa, bunday prizma **muntazam prizma** deyiladi.

$S_{,í} = P_{asos} \cdot l$ - to‘g‘ri prizma yon sirtini hisoblash formulasi, bunda P_{asos} - asos perimatri, l - yon qirralari uzunligi. $S_T = S_{yon} + 2S_{as}$,

$$V = S_{as} \cdot H = \frac{1}{3} S_T \cdot r = S_{kes} \cdot l,$$

bunda $S_{êåñ}$ - perpendikulyar kesim yuzi.

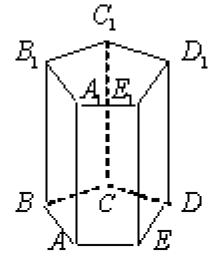
Diagonallari soni: $n(n-3)$.



Ko’pyoqlar

Prizma

1) to’g’ri prizma



Yon sirti: $S_{yon} = P_{as} \cdot H$

To’la sirti: $S_T = 2S_{as} + S_{yon}$;

Hajmi: $V = S_{as} \cdot H$;

n-asosining tomonlarini soni bo’lsa;

Asos diagonallari soni: $\dfrac{n(n-3)}{2}$

Prizma diagonallari soni: $n(n-3)$ ta

Yon qirralarining soni: n ta

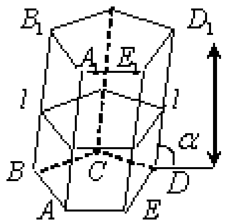
Jami yoqlarining soni: $n+2$ ta

Jami qirralarining soni: $3n$ ta

Jami uchlarining soni: $2n$ ta

Og’ma prizma

$H = l \cdot \sin\alpha$



α-yon qirra va asos tekisligi orasidagi burchak

$S_{yon} = P_{as} \cdot l$; $S_T = 2S_{as} + S_{yon}$; $V = S_{as} \cdot H$; $V = S_{kes} \cdot l$

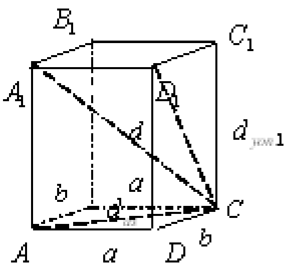
Parallelepiped

1. to’g’ri burchakli parallelepiped.

Asosi to’g’ri to’rtburchak va yon qirrasi asosiga perpendikulyar:

$d_{as} = a^2 + b^2$; $d_{yon.1} = a^2 + c^2$; $d_{yon.2} = b^2 + c^2$; $\sin\alpha = \dfrac{c}{d}$; $S_{as} = ab$; $S_{yon} = 2(a+b)c$;

$S_T = 2S_{as} + S_{yon} = 2(ab + ac + bc)$; $V = S_{as} \cdot H$; $V = abc$

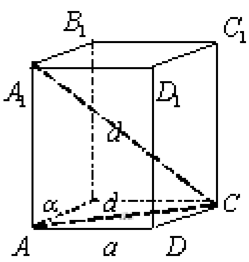


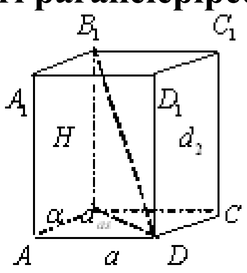
α- parallelepiped diagonali va asos tekisligi orasidagi burchak;

β- parallelepiped diagonali va yon yoq orasidagi burchak;

*5 ta simmetriya tekisligiga ega

II) To’g’ri parallelepiped





$d_{as.1}$, $d_{as.2}$- parallelepiped asosining katta va kichik diagonallari;

d_1, d_2- parallelepipedning katta va kichik asoslari;

γ_1, γ_2- parallelepipedning katta va kichik diagonallarining asos tekisligi bilan hosil qilgan burchaklari;

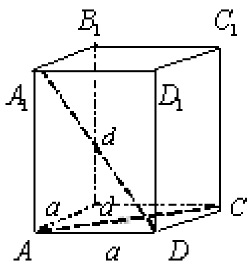
$V = S_{as} \cdot H$; α-o’tkir burchak.

$S_{as} = ab\sin\alpha = \dfrac{d_{as.1} \cdot d_{as.2}}{2}\sin\varphi$; φ- parallelepiped diagonali orasidagi burchak;

$H^2+d_{as.1}^2=d_1^2$; $H^2+d_{as.2}^2=d_2^2$; $\sin\gamma_1=\dfrac{H}{d_1}$; $\cos\gamma_1=\dfrac{d_{as.2}}{d_1}$; $tg\gamma_1=\dfrac{H}{d_{as.1}}$;

$d_{as.1}^2=a^2+b^2+2ab\cos\alpha$; $d_{as.2}^2=a^2+b^2-2ab\cos\alpha$.

Kub



$S_{as}=a^2$; $S_{yon}=4a^2$; $S_T=6a^2$; $V=a^3$; $d_{as}=\sqrt{2}a$; $d=\sqrt{3}a$; $r=\dfrac{a}{2}$;

$R=\dfrac{a\sqrt{3}}{2}$;

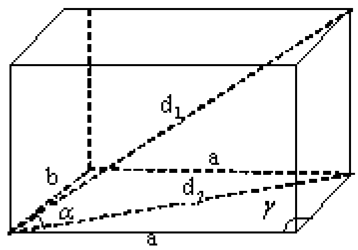
r va R-kubga ichki va tashqi chizilgan sharlarning radiuslari.
*Kub 9 ta simmetriya tekisligiga ega.

PARALLELEPIPED

Prizmaning asosi parallelogramm bo'lsa, bunday prizma **parallelepiped** deyildi. Parallelepipedning hamma yoqlari parallelogrammlardir.

Asosi to'g'ri to'rtburchakdan iborat to'g'ri parallelepiped **to'g'ri burchakli parallelepiped** deyiladi. To'g'ri burchakli parallelepipedning hamma yoqlari to'g'ri to'rtburchaklardan iborat.

To'g'ri burchakli parallelepipedning parallel bo'lmagan qirralarining uzunliklari uning **chiziqli o'lchovlari** deyiladi.



a, b, c - qirralari (o'lchamlari), d_1 - diagonal, d_2 - asosining diagonali.

$S_{as}=ab\sin\gamma$, $S_{yon}=P\cdot c$, $S_{to'la}=S_{yon}+2S_{\text{àñ}}$

$\text{ñ}=\sqrt{a^2+b^2}\cdot tg\alpha$, $V=abc\sin\gamma=S_{\text{àñ}}\cdot H=S_{\text{àñ}}\cdot c=$

$=S_{\perp}\cdot c$; $(S_{\perp}-\text{perpendikulyar kesim yuzi})$

To'g'ri burchakli parallelepiped uchun quyidagilar o'rinli:

$d_1^2=a^2+b^2+c^2$, $d_2^2=a^2+b^2$,

$S_{as}=ab$, $V=abc$.

Kichik diagonal va asos tekisligi orasidagi burchak -

$$\cos\alpha=\frac{\sqrt{a^2+b^2}}{d_1}=\sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}}.$$

To'g'ri parallelepiped uchun quyidagilar o'rinli:

$d_1=\sqrt{a^2+b^2+c^2-2ab\cos\gamma}$; $d_2=\sqrt{a^2+b^2-2ab\cos\gamma}$; $V=S_T\cdot r$.

Kichik diagonal va asos tekisligi orasidagi burchak -

$$\cos\alpha=\frac{\sqrt{a^2+b^2-2ab\cos\gamma}}{d_1}=\sqrt{\frac{a^2+b^2-2ab\cos\gamma}{a^2+b^2+c^2-2ab\cos\gamma}}.$$

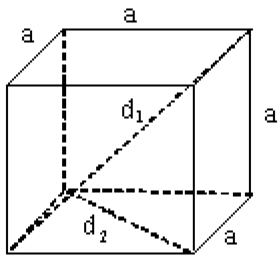
Katta diagonal va asos tekisligi orasidagi burchak -

$$\cos\alpha=\sqrt{\frac{a^2+b^2-2ab\cos\gamma}{a^2+b^2+c^2+2ab\cos\gamma}}=\sqrt{\frac{a^2+b^2-2ab\cos\gamma}{d_1^2+2ab\cos\gamma}}$$

Parallelepiped 12 ta qirra, 6 ta tomon, 8 ta uch va 5 ta simmetriya tekisligiga ega.

KUB

Hamma qirralri teng bo‘lgan to‘g‘ri burchakli parallelepiped **kub** deyiladi.
12 ta qirra, 6 ta yoqqa,
8 ta uchga va 9 ta simmetriya tekisligiga ega.



$d_2 = \sqrt{2}a = \sqrt{d_1^2 - a^2}$, $\quad d_1 = 2R = \sqrt{3}a$.

$S_{as} = a^2$ - asos yuzi.

$S_1 = 6a^2$ - to‘la sirti.

$S_{,í} = 4à^2$ - yon sirti.

$V = a^3$ - hajmi.

$R = \left(\sqrt{3}a\right)/2$ - tashqi chizilgan sfera radiusi.

$r = a/2$ -ichki chizilgan sfera radiusi.

PIRAMIDA

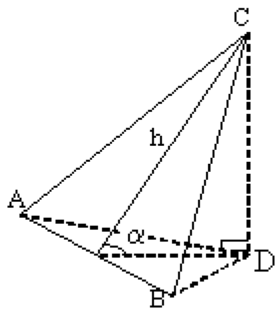
Piramida deb shunday ko‘pyoqqa aytiladiki, u yassi ko‘pburchak – piramida asosidan, asos tekisligida yotmagan nuqta – piramida uchidan va uchni asosining nuqtalari bilan tutashtiruvchi hamma kesmalardan iborat.

Piramidaning uchini asosining uchlari bilan tutashtiruvchi kesmalar piramidaning yon qirralari deyiladi.

Piramidaning sirti asosidan va yon yoqlaridan iborat. Har bir yon yoq uchburchak.

Piramidaning uchidan asos tekisligiga tushirilgan perpendikulyar piramidaning balandligi deyiladi.

Quyidagi formulalarda piramidaning balandligini H bilan belgilaymiz.



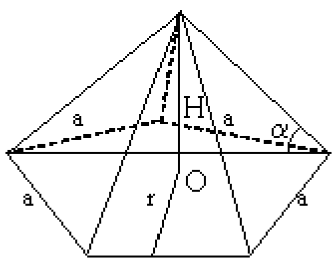
1. Ixtiyoriy piramida:

$S_T = S_{yon} + S_{as}$.

$V = \frac{1}{3} S_{as} \cdot H = \frac{r}{3} \cdot S_T$.

Piramida qirralari soni n ta bo‘lsa, yon yog‘i $-n$ ta, tomonlari – $n+1$ ta.

2. Muntazam piramida.



Peramidaning asosi muntazam ko‘pburchak va balandligining asosi ko‘pburchakning markazi bilan ustma-ust tushsa, bunday piramida muntazam piramida deyiladi. Muntazam piramidaning balandligi yotgan to‘g‘ri chiziq uning o‘qi deyiladi.

Muntazam piramida yon yog‘ining uchidan o‘tkazilgan balandligi apofema deyiladi. Piramida yon yoqlari yuzlarining yig‘indisi uning yon sirti deyiladi.

l - yon qirrasi, f - apofema, a - asosining tomoni, α - yon yog‘i bilan asos tekisligi orasidagi burchak.

$$P_{as}=n\cdot a,\quad S_{as}=\frac{1}{2}n\cdot a\cdot r,$$

$$S_{yon}=\frac{1}{2}P_{as}\cdot f=\frac{S_{asos}}{\cos\alpha},\qquad S_T=S_{yon}+S_{as},\qquad V=\frac{1}{3}S_{as}\cdot H,$$

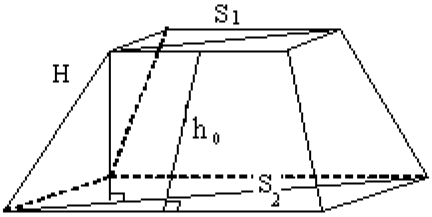
$$l^2=R^2+H^2,\qquad f^2=r^2+H^2.$$

a. Agar piramida yon yoqlari asos tekisligi bilan bir xil burchak hosil qilsa, piramidaning balandligi asosiga ichki chizilgan aylana markaziga tushadi.

b. Agar piramida yon qirralari asos tekisligi bilan bir xil burchak hosil qilsa, piramidaning balandligi asosiga tashqi chizilgan aylana markaziga tushadi.

c. ΔABC ning yuzini tekislikka proyeksiyanlasa, ΔABD hosil bo'ladi: $S_{\Delta ABD}=S_{\Delta ABC}\cdot\cos\alpha$.

3. Kesik piramida.



$$S_{yon}=\frac{1}{2}(P_1+P_2)\cdot h_0,\qquad S_T=S_{yon}+S_1+S_2.$$

$$V=\frac{1}{3}S_T\cdot r=\frac{1}{3}H\left(S_1+\sqrt{S_1\cdot S_2}+S_2\right)$$

4. Muntazam uchburchakli piramida.

l - yon qirrasi, f - apofema, a - asosining tomoni,

α - asosidagi ikki yoqli burchak.

$$f=\sqrt{r^2+H^2}=\sqrt{\frac{a^2}{12}+H^2};\qquad l=\sqrt{R^2+H^2}=\sqrt{\frac{a^2}{3}+H^2};\qquad r=\frac{a\sqrt{3}}{6};\qquad R=\frac{a\sqrt{3}}{3};$$

$$S_{as}=\frac{a^2\sqrt{3}}{4};\qquad S_{yon}=\frac{3}{2}a\cdot f;\quad S_T=\frac{a^2\sqrt{3}}{4}\left(a+\sqrt{a^2+12H^2}\right);\qquad V=\frac{1}{3}S_T\cdot H=\frac{a^2\sqrt{3}}{12}\cdot H;$$

5. Muntazam to'rtburchakli piramida.

l - yon qirrasi, f - apofema, a - asosining tomoni,

α - asosidagi ikki yoqli burchak.

$$f=\sqrt{r^2+H^2}=\sqrt{\frac{a^2}{4}+H^2};\qquad l=\sqrt{R^2+H^2}=\sqrt{\frac{a^2}{2}+H^2};\qquad r=\frac{a}{2};\qquad R=\frac{a\sqrt{3}}{2};$$

$$S_{as}=a^2;\quad S_{yon}=2a\cdot f=\frac{S_{as}}{\cos\alpha};\qquad V=\frac{1}{3}a^2\cdot H;\qquad S_T=\frac{1}{2}p\cdot f+S_{as}$$

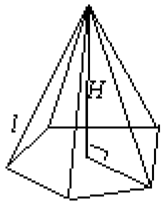
6. Qirrasi a – ga teng bo'lgan muntazam tetraedr.

a - qirrasi.

$$R=\frac{a\sqrt{6}}{4};\qquad r=\frac{a\sqrt{6}}{12};\qquad R=3r;\qquad S_{yon}=\frac{3\sqrt{3}a^2}{4};\qquad S_T=a^2\sqrt{3};\qquad V=\frac{a^2\sqrt{2}}{12}.$$

7. a) Agar r- piramidaga ichki chizilgan sharning, R - esa uning asosiga ichki chizilgan aylananing radiusi bo'lsa, $r=\frac{\sin\alpha\cdot R}{1+\cos\alpha}$ o'rinli. (α - asos va qirrasi orasidagi burchak).

b) Radiusi R –ga teng bo‘lgan sferaga asosi kvadratdan (tomoni –a) iborat bo‘lgan muntazam piramida ichki chizilgan bo‘lsa, uning qirrasi (q) quyidagiga teng: $q = \sqrt{2HR}$.



Piramida

Ixtiyoriy piramida

$S_T = S_{as} + S_{yon}$;

$V = \frac{1}{3} S_{as} \cdot H$;

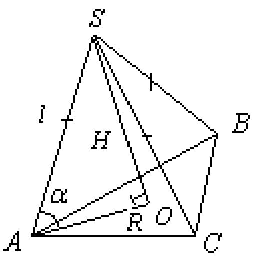
$V = \frac{1}{3} S_T \cdot r_{sh}$

r_{sh} -piramidaga ichki chizilgan sharning radiusi.

Masala yechishda ko’p hollarda piramidaning balandligi asosining qaysi nuqtasiga tushishi muhimdir.

Bunday hollarda quyidagi ma’lumotlardan foydalanish qulayroq:

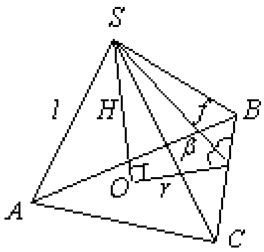
1-hol. Masalada piramidaning barcha yon qirralari teng, yoki piramidaning barcha yon qirralari asos tekisligi bilan bir xil (α) burchak hosil qiladi deyilsa, uning balandligi asosiga tashqi chizilgan aylana markaziga tushadi



$H^2 + R^2 = l^2$; $\sin\alpha = \frac{H}{l}$; $\cos\alpha = \frac{R}{l}$ $tg\alpha = \frac{H}{R}$.

Agar biror ko’pburchakka tashqi aylana chiziq mumkin bo’lmasa, bu ko’pburchak bunday piramidaga asos bo’lolmaydi. ($M - n$ romb)

2-hol. Piramidaning asosidagi barcha ikki yoqli burchaklari (β) teng, yoki piramidaning uchi uning tomonlaridan bir xil uzoqlikda bo’lsa, uning balandligi asosiga ichki chizilgan aylana markaziga tushadi.



$H^2 + r^2 = f^2$; $\sin\beta = \frac{H}{f}$; $tg\beta = \frac{H}{r}$

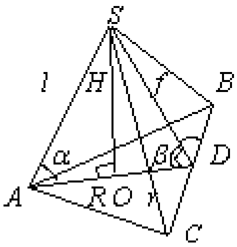
$\cos\beta = \frac{r}{f} = \frac{S_{as}}{S_{yon}}$; $S_{yon} = \frac{P_{as} \cdot f}{2}$;

f -apofema;

P_{as} -piramida asosining perimetri (To’g’ri to’rtburchak asos bo’lolmaydi)

Muntazam piramidalar

1) Muntazam uchburchakli piramida



$R = \frac{a}{\sqrt{3}}$; $r = \frac{a}{2\sqrt{3}}$; $r = \frac{R}{2}$;

f -apofema;

l -yon qirra;

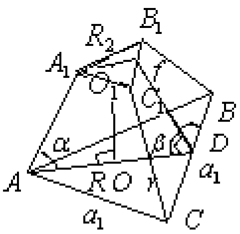
β -yon qirra bilan asos tekisligi orasidagi burchak, yoki yon yoq bilan asos tekisligi orasidagi burchak

$S_{as}=\dfrac{a^2\sqrt{3}}{4}$; $S_{yon}=\dfrac{1}{2}Pf=\dfrac{3}{2}af$; $S_T=S_{as}+S_{yon}$; $V=\dfrac{1}{3}S_{as}H$; $H^2+R^2=l^2$;

$H^2+r^2=f^2$; $H^2=l^2-R^2=f^2-r^2$; $\sin\alpha=\dfrac{H}{l}$; $\cos\alpha=\dfrac{R}{l}$; $tg\alpha=\dfrac{H}{R}$; $\sin\beta=\dfrac{H}{f}$;

$\cos\beta=\dfrac{r}{f}$; $tg\beta=\dfrac{H}{r}$.

Muntazam uchburchakli kesik piramida



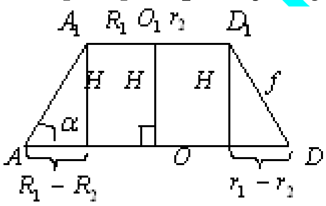
$S_1=\dfrac{a_1^2\sqrt{3}}{4}$; $S_1=\dfrac{a_2^2\sqrt{3}}{4}$; $S_{yon}=\dfrac{1}{2}(P_1+P_2)f=\dfrac{3}{2}(a_1+a_2)f$;

$S_T=S_1+S_2+S_{yon}$; $V=\dfrac{1}{3}H(S_1+\sqrt{S_1S_2}+S_2)$; $\dfrac{a_1}{a_2}=\dfrac{R_1}{R_2}=\dfrac{r_1}{r_2}$;

$\dfrac{S_1}{S_2}=\dfrac{a_1^2}{a_2^2}=\dfrac{R_1^2}{R_2^2}=\dfrac{r_1^2}{r_2^2}$;

S_1, S_2 -piramida asoslarining yuzlari.

* AA_1DD_1 trapetsiya ajratib olinganda



$\sin\alpha=\dfrac{H}{l}$; $\cos\alpha=\dfrac{R_1-R_2}{l}$; $tg\alpha=\dfrac{H}{R_1-R_2}$

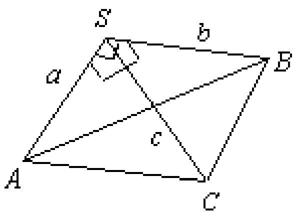
$\sin\beta=\dfrac{H}{f}$; $\cos\beta=\dfrac{r_1-r_2}{f}=\dfrac{S_1-S_2}{S_{yon}}$; $tg\beta=\dfrac{H}{r_1-r_2}$;

$(R_1-R_2)^2+H^2=l^2$;

$(r_1-r_2)^2+H^2=f^2$

$l^2-(R_1-R_2)^2=f^2-(r_1-r_2)^2$.

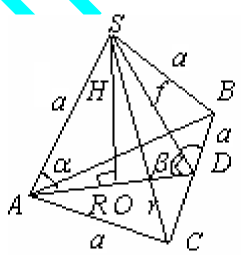
Yon qirrasi o'zaro perpendikulyar bo'lgan uchburchakli piramida



$V=\dfrac{abc}{6}$

$S_{yon}=\dfrac{1}{2}(ab+bc+ac)$.

***Tetraedr-hamma qirralari teng muntazam uchburchakli piramida.**



$H=\dfrac{a\sqrt{6}}{3}$; $S_T=a^2\sqrt{3}$; $V=\dfrac{a^3\sqrt{2}}{12}$

*** Tetraedr asosiga ichki va tashqi chizilgan aylanalar radiuslari**

$r=\dfrac{a}{2\sqrt{3}}=\dfrac{a\sqrt{3}}{6}$; $R=\dfrac{a}{\sqrt{3}}=\dfrac{a\sqrt{3}}{3}$

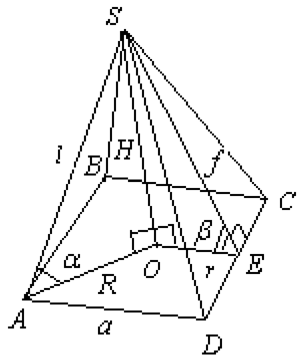
*** Tetraedrga ichki va tashqi chizilgan sharlar radiuslari**

$r_{sh}=\dfrac{a\sqrt{6}}{12}=\dfrac{R_{sh}}{3}$; $R_{sh}=\dfrac{a\sqrt{6}}{4}=3r_{sh}$.

*Yon qirrasi va asos tekisligi orasidagi burchak kosinusi: $\cos\alpha = \frac{1}{\sqrt{3}}$.

* Asosidagi ikki yoqli burchak kosinusi: $\cos\beta = \frac{1}{3}$.

Muntazam to’rtburchakli piramida



$R = \frac{a}{\sqrt{2}}$; $r = \frac{a}{2}$.

α -yon qirra va asos tekisligi orasidagi burchak.

β-asosidagi ikki yoqli (yoki yon yoq va asos tekisligi orasidagi) burchak

$\sin\alpha = \frac{H}{l}$; $\cos\alpha = \frac{R}{l}$; $tg\alpha = \frac{H}{R}$

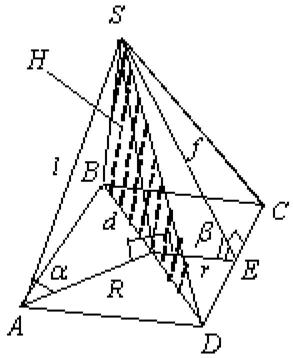
$\sin\beta = \frac{H}{f}$; $\cos\beta = \frac{r}{f}$; $tg\beta = \frac{H}{r}$

$R^2 + H^2 = l^2$; $r^2 + H^2 = f^2$; $l^2 - R^2 = f^2 - r^2$.

$S_{as} = a^2$; $S_{yon} = \frac{1}{2}Pf$; $S_T = S_{as} + S_{yon}$.

$V = \frac{1}{3}S_{as}H$; $V = \frac{1}{3}S_T \cdot r_{sh}$. r_{sh} -piramidaga ichki chizilgan sharning radiusi.

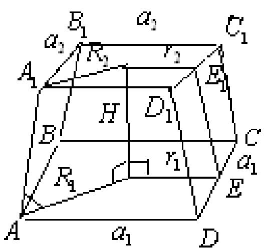
Diagonal kesim



$d = \sqrt{2}a$; $d = \frac{2}{\cos\alpha}$

$S_k = \frac{d \cdot H}{2}$; $H = \frac{a\sqrt{2}}{2}tg\alpha$; $H = l \cdot \sin\alpha$

Muntazam to’rtburchakli kesik piramida

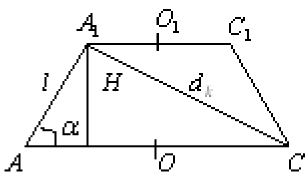


$\sin\alpha = \frac{H}{l}$; $\cos\alpha = \frac{R_1 - R_2}{l}$; $tg\alpha = \frac{H}{R_1 - R_2}$;

$\sin\beta = \frac{H}{f}$; $\cos\beta = \frac{r_1 - r_2}{f} = \frac{S_1 - S_2}{S_{yon}}$; $S_1 = a_1^2$; $S_2 = a_2^2$; $S_{yon} = \frac{1}{2}(P_1 - P_2)f$; $S_T = S_1 + S_2 + S_{yon}$;

$V = \frac{1}{3}(S_1 + \sqrt{S_1 S_2} + S_2)$; $\frac{a_1}{a_2} = \frac{R_1}{R_2} = \frac{r_1}{r_2}$; $\frac{S_1}{S_2} = \frac{a_1^2}{a_2^2} = \frac{R_1^2}{R_2^2} = \frac{r_1^2}{r_2^2}$.

Diagonal kesimi

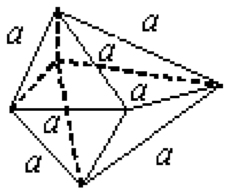


$AC = d_1 = \sqrt{2}a_1$; $A_1C_1 = d_2 = \sqrt{2}a_2$

d_1, d_2-piramida asoslarining diagonallari.

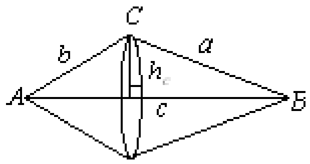
$S_k = \frac{d_1 + d_2}{2}H$; $d_k^2 = \left(\frac{d_1 + d_2}{2}\right)^2 + H^2$.

Oktoedr-muntazam sakkizyoq



$$r_{sh}=\frac{a\sqrt{6}}{6};\ R_{sh}=\frac{a\sqrt{2}}{2};\ S_T=2a^2\sqrt{3};\ V=\frac{a^3\sqrt{2}}{3}$$

Uchburchakni uning biror tomoni atrofida aylantirishdan hosil bo'lgan jismning sirti va hajmi



c-tomon atrofida aylantirganda;

$S_c=\pi\cdot h_c(a+b);\ V_c=\frac{1}{3}\pi\cdot c\cdot h_c^2$

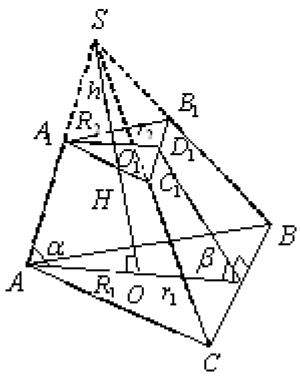
a-tomon atrofida aylantirganda;

$S_a=\pi\cdot h_a(b+c);\ V_a=\frac{1}{3}\pi\cdot a\cdot h_a^2$

b-tomon atrofida aylantirganda;

$S_b=\pi\cdot h_b(a+c);\ V_b=\frac{1}{3}\pi\cdot b\cdot h_b^2$.

To'ldirilgan piramida



$SABC$-katta (yoki to'la) piramida;

$SA_1B_1C_1$-kichik piramida;

H_T- to'la piramidaning balandligi;

H-kesik piramidaning balandligi;

h-kichik piramidaning balandligi;

V_T-to'la piramidaning hajmi;

V_1-kichik piramidaning hajmi;

V-kesik piramidaning hajmi

$$H_T=SO=H+h;\ H=OO_1;\ h=SO_1;\ V_T=V+V_1;$$

$$V=\frac{1}{3}H(S_1+\sqrt{S_1S_2}+S_2);\ V_T=\frac{1}{3}S_1H_T;\ V_1=\frac{1}{3}S_2h;$$

$$\frac{a_1}{a_2}=\frac{H_T}{h}=\frac{H+H}{h}=\frac{R_1}{R_2}=\frac{r_1}{r_2};\ \frac{S_1}{S_2}=\frac{a_1^2}{a_2^2}=\frac{H_T^2}{h^2}=\frac{(H+h)^2}{h^2}=\frac{R_1^2}{R_2^2}=\frac{r_1^2}{r_2^2};$$

$$\frac{V_T}{V_2}=\frac{a_1^3}{a_2^3}=\frac{H_T^3}{h^3}=\frac{(H+h)^3}{h^3}=\frac{R_1^3}{R_2^3}=\frac{r_1^3}{r_2^3}$$

MUNTAZAM KO'PYOQLAR.

Agar qavariq ko'pyoq yoqlarining tomonlari soni bir xil bo'lgan muntazam ko'pburchakdan iborat bo'lsa va shu bilan birga ko'pyoqning har bir uchida bir xil miqdordagi qirralar uchrashsa, bunday qavariq ko'pyoq muntazam ko'pyoq deyiladi.

Muntazam qavariq ko'pyoqlarning besh turi bor: tetraedr, kub, oktaedr, dodekaedr, ikosaedr.

	R	**r**	**$S_{to'la}$**	**V**
Tetraedr(4)	$\frac{a\sqrt{6}}{4}$	$\frac{a\sqrt{6}}{12}$	$a^2\sqrt{3}$	$\frac{a^3\sqrt{2}}{12}$

Kub (6)	$\frac{a\sqrt{3}}{2}$	$\frac{a}{2}$	$6a^2$	a^3
Oktaedr (8)	$\frac{a\sqrt{2}}{2}$	$\frac{a\sqrt{6}}{6}$	$2a^2\sqrt{3}$	$\frac{a^3\sqrt{2}}{3}$
Dodekaedr (12)	$\frac{a\sqrt{3}}{\sqrt{5}-1}$	$\frac{a}{2}\sqrt{\frac{25+11\sqrt{5}}{10}}$	$3a^2\sqrt{25+10\sqrt{5}}$	$\frac{a^3}{4}\left(15+7\sqrt{5}\right)$
Ikosaedr (20)	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	$\frac{a\left(3+\sqrt{5}\right)}{4\sqrt{3}}$	$5a^2\sqrt{3}$	$\frac{5a^3}{12}\left(3+\sqrt{5}\right)$

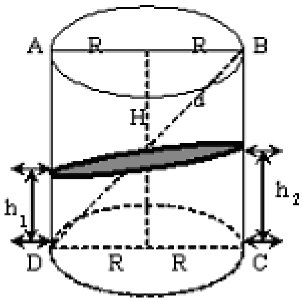
Eyler formulasi: $Y+P=Q+2$,
bu yerda Y- qavariq ko‘pyoqning uchlari soni, P-yoqlari soni, Q- qirrallari soni.

O‘XSHASH KO‘PYOQLAR.

$$\frac{S_1}{S_2}=\left[\frac{P_1}{P_2}\right]^2; \qquad \frac{V_1}{V_2}=\left[\frac{a_1}{a_2}\right]^3=\left[\frac{H_1}{H_2}\right]^3=\left[\frac{P_1}{P_2}\right]^3=\left[\frac{l_1}{l_2}\right]^3.$$

SILINDIR.

Parallel ko‘chirish bilan ustma-ust joylashadigan va bitta teksilikda yotmaydigan ikki doiradan va bu doiralarning mos nuqtalarini tutashtiruvchi hamma parallel to‘g‘ri chiziq kesmalaridan tashkil topgan jism silindr deyiladi. Doiralar silindirning asoslari deyiladi, doira aylanalarining mos nuqtalarini tutashtiruvchi kesmalar silindirning yasovchilari deyiladi.



Silindir asosining radiusi silindirning radiusi deyiladi.

Silindir asoslarining tekisliklari orasidagi masofa silindirning balandligi deyiladi. Asoslarining markazidan o‘tuvchi to‘g‘ri chiziq silindirning o‘qi deyiladi. Bu o‘q yasovchilarga parallel bo‘ladi.

$$S_{as}=\pi\cdot R^2; \qquad S_{yon}=2\pi\cdot RH; \qquad S_T=2\pi R(R+H)=2\pi RH+2\pi R^2.$$

$$S_T=S_{as}+2S_{yon}; \quad S_{kes}=S_{ABCD}=2RH-o'q\ kesim.$$

$$V=\pi\ R^2\ H.$$

Agar ABCD kvadrat bo‘lsa –

$$H=2R; \quad R=H/2; \quad S_{kes}=S_{ABCD}=4R^2=H^2=2RH;$$

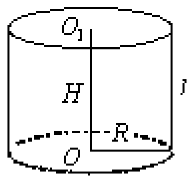
$$S_T=6\pi\ R^2=\frac{3}{2}\pi\ H^2;$$

$$V=2\pi\ R^2=\frac{1}{4}\pi\ H^2=\frac{\pi\cdot d^2}{4}\cdot H=S_{ABCD}\pi\ R; \quad (d=DB)$$

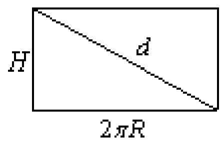
$$V=\pi\ R^2\left(\frac{h_1+h_2}{2}\right).$$

Agar $S_{yon}=2S_{as}$ va $S_{o'q\_kes}=G$ bo‘lsa, $V=\frac{\pi\ G(2G)}{4}$ o‘rinli.

SILINDR



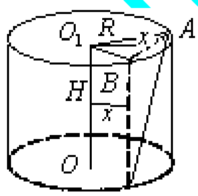
Silindr yoyilmasi

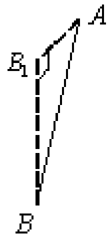


OO_1-silindr o'qi; H-balandligi;
l-yasovchisi; R-asosining radiusi;
S_{as}-silindr asosining yuzi; V-silindr hajmi.

$OO_1 = H = l$; $S_{as} = \pi R^2$; $S_{yon} = 2\pi RH$; $S_T = 2S_{as} + S_{yon} = 2\pi R(R+H)$;
$V = S_{as}H = \pi R^2 H$.

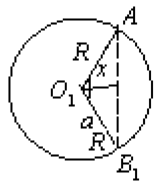
Silindr va kesim





$AB_1^2 + H^2 = AB^2$; $x^2 + \left(\frac{AB_1}{2}\right)^2 = R^2$

$\sin\frac{\alpha}{2} = \frac{AB_1}{2R}$; $\cos\frac{\alpha}{2} = \frac{x}{R}$; $tg\frac{\alpha}{2} = \frac{AB_1}{2x}$



x-kesma va silindr o'qi orasidagi masofa

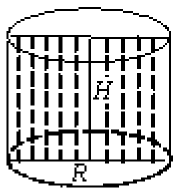
Silindr va kesimlar

Silindrning o'qiga parallel kesimlar

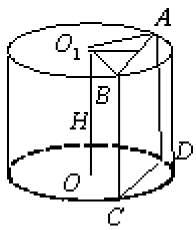
I. O'q kesim.

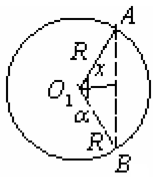
$S_k = 2RH$

Teng tomonli silindrda o'q kesim kvadratdan iborat bo'ladi: $H = 2R$



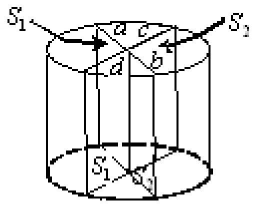
II. Silindr o'qidan biror x masofada o'tuvchi kesim.





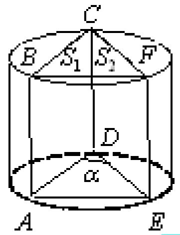
$S_k = Ab \cdot H$; $x^2 + \left(\frac{AB}{2}\right)^2 = R^2$; $\sin\frac{\alpha}{2} = \frac{AB}{2R}$; $\cos\frac{\alpha}{2} = \frac{x}{R}$; $tg\frac{\alpha}{2} = \frac{AB}{2x}$

III. Silindr o'qiga parallel o'zaro kesishuvchi kesimlar



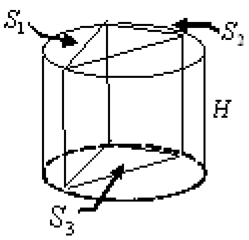
$a \cdot b = c \cdot d = R^2 - x^2$; $S_1 \cdot S_2 = S_2 \cdot S_1$.
Tekisliklar perpendikulyar bo'lsa
$S_{o'q\,k} = \sqrt{S_1^2 + S_2^2 + S_3^2 + S_4^2}$ o'rinli
x-silindr asosining markazidan tekisliklarning kesishish nuqtasigacha bo'lgan masofa.
$S_{o'q\,k}$-silindrning o'q kesimi.

IV. Silindr yasovchisidan o'tuvchi tekisliklar

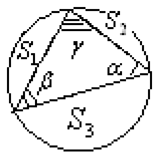


S_{ABFC} - $ABFE$ nuqtalardan o'tuvchi tekislik
$\alpha = 90^\circ$ da, ya'ni $S_1 \perp S_2$ bo'lganda
S_{ABFE}-o'q kesim bo'ladi.
$S_{ABFE}^2 = S_1^2 + S_2^2 - 2S_1S_2\cos\alpha$; $S_{ABFE} = \sqrt{S_1^2 + S_2^2 - 2S_1S_2\cos\alpha}$

Qo'shimcha.



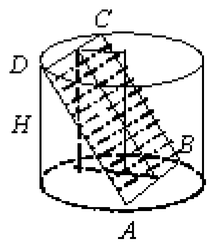
Yuqoridan ko'rinishi



$$\frac{S_1}{\sin\alpha} = \frac{S_2}{\sin\beta} = \frac{S_3}{\sin\varphi} = 2RH$$

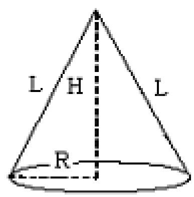
Silindr o'qini kesib o'tuvchi kesim

Kesim $ABCD$ trapetsiyadan iborat bo'ladi.



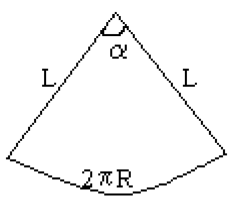
KONUS

Konus deb shunday jismga aytiladiki, u doira – konus asosidan, shu doira tekisligida yotmagan nuqta – konus uchidan va konusning uchini asosining hamma nuqtalari bilan tutashtiruvchi kesmalardan iborat bo'ladi. Konus uchini asos aylanasi nuqtalari bilan tutashtiruvchi kesmalar konusning yasovchilari deyiladi. Konus sirti asosidan va yon sirtidan iborat.



Konusning uchi bilan asos aylanasining markazini tutashtiruvchi to'g'ri chiziq asos tekisligiga perpendikulyar bo'lsa, bunday konus to'g'ri konus deyiladi.

Konusning uchidan uning asosiga tushirilgan perpendikulyar konusning balanndligi deyiladi.



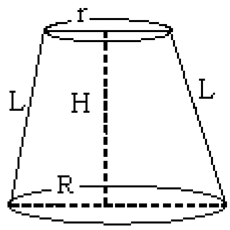
$L = \sqrt{R^2 + H^2}$, $S_{as} = \pi R^2$, $S_{yon} = \pi RL$.

$S_T = \pi R(L + R) = S_{yon} + S_{as}$.

$V = \frac{1}{3}\pi \cdot R^2 \cdot H = S_T \cdot r$, $R = \frac{\sqrt{2}}{2} L$.

Yoyilmasi uchidagi burchak: $\alpha = \frac{2R}{L} 180^0$.

KESIK KONUS



$L=\sqrt{(R-r)^2+H^2}$, $S_{yon}=\pi\ L(R+r)$, $S_{o'q\_kes}=(R+r)H$.

$S_T=S_{yon}+S_1+S_2=S_{yon}+\pi\left(R^2+r^2\right)=\pi\ (R+r)L+\pi r^2+\pi\ R^2$.

$V=\frac{1}{3}\pi\ H\left(R^2+Rr+r^2\right)$.

$\frac{V_1}{V}=\left(\frac{H_1}{H}\right)^3$, $\frac{S_1}{S}=\left(\frac{H_1}{H}\right)^2$.

Xususiy hollar.

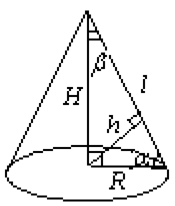
1) Agar $L=2R$ bo'lsa, $S_T=9\ \pi\ r^2$ (r – ichki chizilgan sfera radiusi.) bo'ladi.

2) $L^2=2HR_1$, $R_1=\frac{R^2+H^2}{2H}$, $L=\sqrt{2}R$, $V=\frac{1}{3}\pi\cdot R_1^2$, bunda R_1 - konusga tashqi chizilgan sfera radiusi.

3) $r_1=\frac{\sin\alpha\cdot R}{1+\cos\alpha}=\frac{R\cdot H}{L+R}$ bunda r_1 - konusga ichki chizilgan sfera radiusi, α - yasovchi va asos orasidagi burchak.

4) Kesik konus uchun $L=R+r$; $H=2r_1$ bunda r_1 - konusga ichki chizilgan sfera radiusi.

R-konus asosining radiusi;
l-konus yasovchisi;
H-balandligi;
h-konus asosi markazidan uning yasovchisigacha bo'lgan masofa;
α-yasovchi va asos tekisligi orasidagi burchak;
β-yasovchi va balandlik orasidagiburchak.

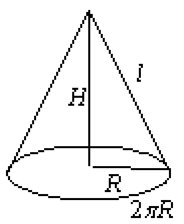


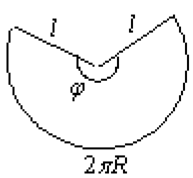
$\sin\alpha=\frac{H}{l}=\frac{h}{R}$; $\cos\alpha=\frac{R}{l}=\frac{S_{as}}{S_{yon}}$; $tg\alpha=\frac{H}{R}$

$H^2+R^2=l^2$; $R\cdot H=l\cdot h$; $S_{as}=\pi R^2$; $S_{yon}=\pi Rl$;

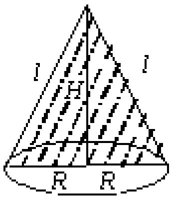
$S_T=S_{as}+S_{yon}=\pi R(R+l)$; $V=\frac{1}{3}S_{as}H=\frac{1}{3}\pi R^2H=\frac{1}{3}S_{yon}h$

Konusning yoyilmasi





$2\pi R=l\cdot\varphi$; $\varphi=\frac{2\pi R}{l}$

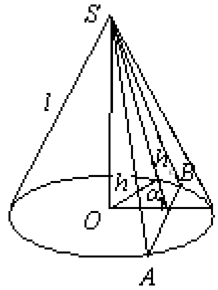


Konusning o'q kesimi

$S_k = RH$

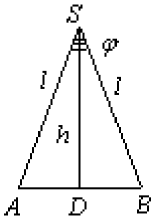
Teng tomonli konusning o'q kesimi muntazam uchburchakdan iborat bo'ladi.

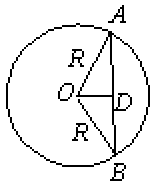
Konusning yasovchilari orqali o'tkazilgan kesim



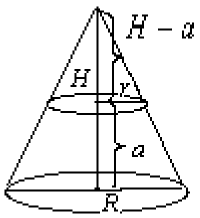
h_k -kesimning balandligi

h -konus asosining markazidan kesimgacha bo'lgan masofa;





Konusning asosiga parallel qilib o'tkazilgan kesim



$$\frac{S_{as}}{S_k} = \frac{R^2}{r_k^2} = \frac{H^2}{(H-a)^2}$$

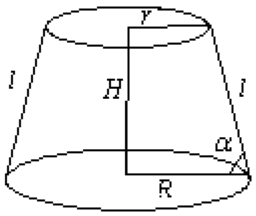
$$\frac{R}{r_k} = \frac{H}{H-a}$$

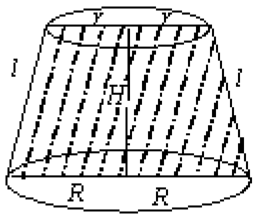
V -konusning to'liq hajmi;

V_k -konusning kesimdan yuqori qismining hajmi;

$\frac{V}{V_k} = \frac{R^3}{r_k^3} = \frac{H^3}{(H-a)^3}$; a -konus asosidan kesimgacha bo'lgan masofa.

Kesik konus





$H^2 + (R-r)^2 = l^2$; $\sin\alpha = \frac{H}{l}$; $tg\alpha = \frac{H}{R-r}$; $\cos\alpha = \frac{R-r}{l} = \frac{S_{as\,1} - S_{as\,2}}{S_{yon}}$; $S_{as\,1} = \pi R^2$;

$S_{as\,2} = \pi r^2$; $S_{yon} = \pi l(R+r)$; $S_T = S_{as\,1} + S_{as\,2} + S_{yon}$; $S_T = \pi(R^2 + r^2 l(R+r))$;

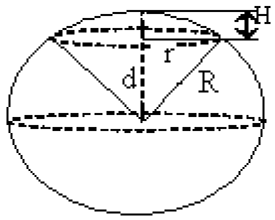
$V = \frac{1}{3}H(S_{as\,1} + \sqrt{S_{as\,1}S_{as\,2}} + S_{as\,2})$; $V = \frac{1}{3}\pi H(R^2 + Rr + r^2)$; $S_{o'q\,k} = (R+r)H$

SHAR

Fazoning berilgan nuqtadan berilgan masofadan katta bo'lmagan uzoqlikda yotgan hamma nuqtalaridan iborat jism **shar** deyiladi. Berilgan nuqta **sharning markazi**, berilgan masofa esa **sharning radiusi** deyiladi.

Sharning chegarasi shar sirti yoki **sfera** deb ataladi.

Shar sirtining ikki nuqtasini tutashtiruvchi va sharning markazidan o'tuvchi kesma **diametr** deyiladi. Istalgan diametrning oxirlari **sharning diametral qarama-qarshi nuqtalari** deyiladi.



$R^2 = d^2 + r^2$, $d^2 = R^2 - r^2$, $S_{shar\ sirti} = 4\pi R^2$, $V = \frac{4}{3}\pi R^3 = \frac{\pi}{6}d^3$.

Shar sektori: $S_{to'la.sek} = 2\pi R(2H + r)$, $V_{sek} = \frac{3}{2}\pi R^2 \cdot H = \frac{\pi}{6}d^2 H$.

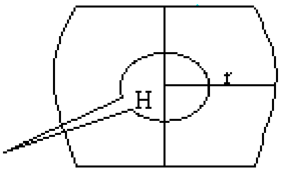
Shar segmenti:

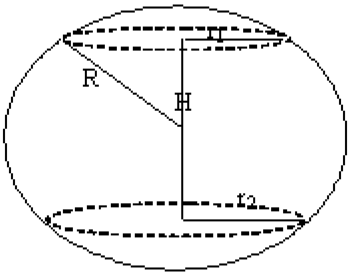
$r = \sqrt{H(2R - H)}$, $S_{yon} = 2\pi RH = \pi(r^2 + H^2)$, $S_{to'la.seg} = \pi(2r^2 + H^2)$,

$V_{seg} = \pi H^2\left(R - \frac{1}{3}H\right) = \frac{1}{6}\pi H(3r^2 + H^2)$, $0 < H \le R$.

SHAR KAMARI (HALQASI)

$S_{kamar} = 2\pi\ RH$; $S_{to'la\ kamar} = \pi\ (2RH + r_1^2 + r_2^2)$;





$V_{kamar} = \frac{1}{6}\pi\ H(3r_1^2 + 3r_2^2 + H^2)$, $0 \le H \le R$;

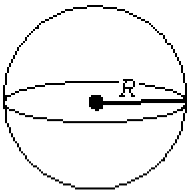
$V_{kamar} = \pi\ H\left[r + \frac{2}{3}(R - r)\right]^2$.

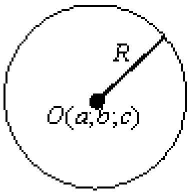
Markazi koordinata boshida, radiusi R bo'lgan sfera tenglamasi: $x^2 + y^2 + z^2 = R^2$

Markazi $A(a,b,c)$ nuqtada, radiusi R bo'lgan sfera tenglamasi:

$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$

SHAR





$S = 4\pi R^2 = \pi\, d^2$; $V = \frac{4}{3}\pi R^3 = \frac{\pi\, d^3}{6}$

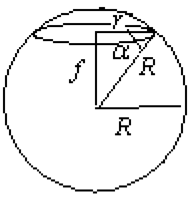
*Markazi $O(a;b;c)$ nuqtada bo'lgan R radiusli shar tenglamasi:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \le R^2$$

*Markazi $O(a;b;c)$ nuqtada bo'lgan R radiusli sfera tenglamasi:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

Shar kesimi. Shar va tekislik



R, r -shar va kesim radisulari;

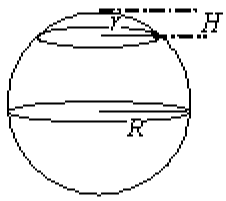
f -shar markazidan kesimgacha masofa;

α -sharning radisui va tekislik orasidagi burchak

$f^2 + r^2 = R^2$; $S_k = \pi r^2$; $\sin\alpha = \frac{f}{R}$; $\cos\alpha = \frac{r}{R}$; $tg\alpha = \frac{f}{r}$.

*Agar biror to'rtburchakning tomonlari ($M - n$ uchburchak, to'rtburchak va hokazo) sharga urinsa, kesim shu ko'pburchakka ichki chizilgan doira bo'ladi va r ko'pburchakka ichki chizilgan doiraning radiusi bo'ladi.

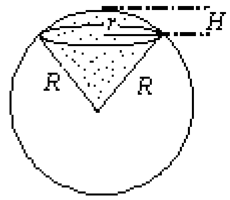
Shar segmenti



r -shar segmenti asosining radiusi;
H -shar segmenti balandligi;
S_{yon} -segment yon sirti;
S_T -to'la sirti;
V -segment hajmi;

$r=\sqrt{H(2R-H)}$; $S_{yon}=2\pi RH$; $S_T=2\pi RH+\pi r^2$; $V=\frac{1}{3}\pi H^2(3R+H)$.

Shar sektori

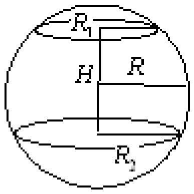


$S_T=\pi R(2H+r)$; $V=\frac{2\pi}{3}R^2H$; $V=\frac{\pi}{6}d^2H$

d -sharning diametri

Shar halqasi

H -shar halqasi balandligi;
R_1, R_2 -shar halqasining radiuslari



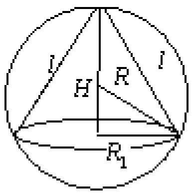
$S_{yon}=2\pi RH$; $S_T=\pi(2RH+R_1^2+R_2^2)$;

$H=\sqrt{R^2-R_1^2}+\sqrt{R^2-R_2^2}$; $V=\frac{1}{6}\pi H(3R_1^2+3R_2^2+H^2)$;

$V=\frac{1}{6}\pi H^2+\frac{1}{2}\pi(R_1^2+R_2^2)$.

Eslatma. R_1 va R_2 -raiuslar shar markazidan bir tomonga joylashgan bo'lsa, $H=\left|\sqrt{R^2-R_1^2}-\sqrt{R^2-R_2^2}\right|$ formula o'rinli bo'ladi.

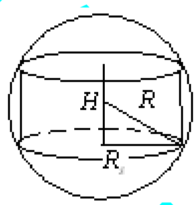
Sharga ichki chizilgan konus



$(H-R)^2+R_1^2=R^2$; $R=\frac{R_1^2+H^2}{2H}$.

R -shar radiusi;
H -konus balanligi;
R_1 -konus asosining radiusi.

Sharga ichki chizilgan silindr



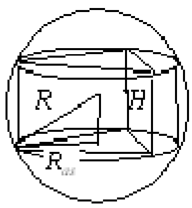
R_s -silinr asosining radiusi

$R_s^k+\frac{H^2}{4}=R^2$

$H=\frac{2\sqrt{3}}{3}R$ bo'lganda sharga ichki chizilgan silindrning hajmi eng katta bo'ladi.

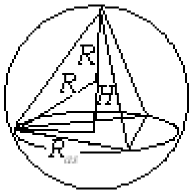
Sharga ichki chizilgan prizma

$R_{as}^2+\left(\frac{H}{2}\right)^2=R^2$



R_{as} -prizma asosiga tashqi chizilgan aylana radiusi.

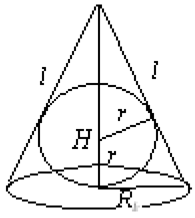
Sharga ichki chizilgan piramida



$$(H-R)^2+R_1^2=R^2$$

$$R=\frac{R_{as}^2+H}{2H}$$

Sharga tashqi chizilgan konus

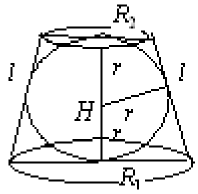


$$\frac{l}{H-r}=\frac{R_k}{r}=\frac{H}{l-R_k};$$

$$\frac{R_k}{r}=\frac{H}{\sqrt{H^2-2Hr}}$$

r-sharning radiusi

Sharga tashqi chizilgan kesik konus

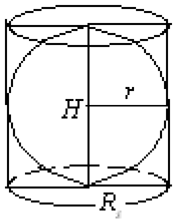


$l=R_1+R_2=l_{o'r}$

$H=2r$; $r=\dfrac{H}{2}$

$l_{o'r}$-trapetsiyaning o'rta chizig'i.

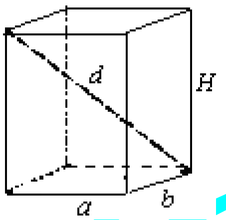
Silindrga ichki chizlgan shar



$$r=R_s=\frac{H}{2}.$$

Silindrning o'q kesimi kvadratdan ibrat bo'ladi.

Sharga va parallelipiped



R_{sh}-tashqi chizilgan sharning radiusi

r_{sh}-ichki chizilgan sharning radiusi

$R_{sh}=\dfrac{d}{2}$; $r_{sh}=\dfrac{H}{2}$; $V=\dfrac{1}{3}S_T\cdot r_{sh}$

Muntazam piramidaga shar ichki chizilganda

$$r_{sh}=\frac{\sin\alpha}{1+\cos\alpha}r_{as}$$

r_{as}-piramida asosiga ichki chizilgan aylananing radiusi

α-piramida asosidagi ikki yoqli burchak.

VEKTORLAR

1) Birlik vektorlar - $i=(1;0)$, $j=(0;1)$; $|i|=|j|=1$; $i\cdot j=0$.
2) Vektorlar ustida amallar:

1. $A(x_1;y_1)$, $B(x_2;y_2)$ bo'lsa, $\vec{AB}(x_2-x_1;y_2-y_1)$. A nuqta vektorning boshi, $\hat{A}$ - nuqta vektorning oxiri deyiladi. Agar $a_1=x_2-x_1$, $a_2=y_2-y_1$ bo'lsa, $\vec{AB}(a_1,a_2)$ yoki $\vec{a}=(a_1,a_2)$ kabi belgilanadi.

2. $|\vec{AB}|=\sqrt{(x_1-x_2)^2+(y_2-y_1)^2}$, $\vec{a}=\sqrt{a_1^2+a_2^2}$, $|\vec{AB}|=|\vec{a}|$.

Agar fazoda $A(x_1;y_1;z_1)$, $B(x_2;y_2;z_2)$ bo‘lsa,

$\left|\vec{AB}\right| = \sqrt{(x_1-x_2)^2+(y_2-y_1)^2+(z_2-z_1)^2}$, $\left|\vec{a}\right| = \sqrt{a_1^2+a_2^2+a_3^2}$ bo‘ladi. Bular vektornig moduli yoki ikki nuqta orasidagi masofa ham deyiladi.

3. $\vec{a}=(a_1,\ a_2)$, $\vec{b}=(b_1,b_2)$ bo‘lsin $\vec{a}\pm\vec{b}=(a_1\pm b_1,\ a_2\pm b_2)$ - vektorlarni qo‘shish yoki ayirish.

4. $n\cdot\vec{a}=(n\cdot a_1,\ n\cdot a_2)$ - vektorlarni songa ko‘paytirish.

5. $\vec{a}\cdot\vec{b}=a_1\cdot b_1+a_2\cdot b_2=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha$ - skalyar ko‘paytmasi. Bunda α - ikki vektor orasidagi burchak. $\left|\vec{a}\right|^2=\sqrt{\vec{a}\cdot\vec{a}}$.

6. $\vec{a}\cdot\vec{b}=0$ bo‘lsa vektorlar perpendikulyar bo‘ladi, ya’ni $\vec{a}\perp\vec{b}$ bo‘lsa, $\alpha=90^0$ bo‘ladi.

7. $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ bo‘lsa, vektorlar kolleniar (parallel) bo‘ladi, ya’ni $\vec{a}\parallel\vec{b}$.

8. Vektorlar orasidagi burchak: $\cos\alpha=\frac{a_1\cdot b_1+a_2\cdot b_2}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}=\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}$.

9. $\vec{AB}(a,b)$ to‘g‘ri chiziqqa (vektorga) perpendikulyar va $(x_0;y_0)$ nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasi: $a(x-x_0)+b(y-y_0)=0$.

NUQTADAN TO’G’RI CHIZIQGACHA BO’LGAN MASOFA

$M(x_0;y_0)$ nuqta va $ax+by+c=0$ to‘g‘ri chiziq berilgan bo‘lsa, nuqtadan to‘g‘ri chiziqgacha masofa (d) quyidagi formula orqali aniqlanadi: $d=\frac{|a\cdot x_0+b\cdot y_0+c|}{\sqrt{a^2+b^2}}$.

KESMANI BERILGAN NISBATDA BO‘LISH

$MA:MB=\lambda$, $M(x_0;y_0)$, $A(x_1;y_1)$, $B(x_2;y_2)$ bo’lsa:

$$x_o=\frac{x_1+\lambda\cdot x_2}{1+\lambda};\qquad y_0=\frac{y_1+\lambda\cdot y_2}{1+\lambda}.$$

$MA=MB$ bo’lsa: $x_0=\frac{x_1+x_2}{2}$; $y_0=\frac{y_1+y_2}{2}$.

IKKI NUQTADAN O‘TGAN TO‘G‘RI CHIZIQ TENGLAMASI.

Tekislikda: $A(x_1,y_1)$, $B(x_2,y_2)$: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$.

Fazoda: $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.

$A(x_1,y_1)$ nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasi: $y-y_1=k(x-x_1)$.

$M(x_0,y_0,z_0)$ nuqtadan $Ax+By+Cz+D=0$ tekislikkacha masofa (d) quyidagi formula orqali topiladi: $d=\frac{|A\cdot x_0+B\cdot y_0+C\cdot z_o+D|}{\sqrt{A^2+B^2+C^2}}$.

$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ tekisliklar orasidagi burchak quyidagi formula bilan topiladi:

$$\cos\alpha = \frac{A_1 \cdot A_2 + B_1 \cdot B_2 + C_1 \cdot C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

Agar tekisliklar parallel bo‘lsa, ular orasidagi masofa (d) quyidagi formula orqali topiladi:

$$d = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}.$$

$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$ bo‘lsa, $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ tekisliklar parallel bo‘ladi.

SIMMETRIYA

$A(x;y)$ nuqtaga Ox o‘qiga nisbatan simmetrik nuqta - $A_1(x;-y)$, Oy o‘qiga nisbatan - $A_2(-x;y)$, koordinata boshiga nisbatan- $A_0(-x;-y)$ bo‘ladi.

$A(x,y,z)$ nuqtaga Ox o‘qiga nisbatan simmetrik nuqta - $A_1(x;-y;-z)$, Oy o‘qiga nisbatan - $\grave{A}_2(-x;y;-z)$, Oz o‘qiga nisbatan - $A_3(-x;-y;z)$, Oxy (XY) tekisligiga nisbatan- $A_4(x;y;-z)$, Oxz (XZ) tekisligiga nisbatan - $A_5(x;-y;z)$, Oyz (YZ) tekisligiga nisbatan - $A_6(-x;y;z)$, koordinata boshiga nisbatan- $A_0(-x;-y;-z)$ bo‘ladi.

Hech qaysi 3 tasi bir to‘g‘ri chiziqda yotmaydigan n ta nuqtani tutashtirishdan $\frac{n(n-1)}{2}$ ta to‘g‘ri chiziq, $\frac{n(n-1)(n-2)}{6}$ ta tekislik hosil bo‘ladi.

TO‘G‘RI CHIZIQ VA TEKISLIKKA OID BA’ZI MULOHAZALAR

1. Agar ikki to‘g‘ri chiziqni uchinchi to‘g‘ri chiziq kesib o‘tganda, bir tomondagi tashqi burchaklar yig‘indisi 180^0ga teng bo‘lsa, bu to‘g‘ri chiziqlar paralleldir.
2. Agar tekislikda yotmagan to‘g‘ri chiziqlar tekislikka perpendikulyar bo‘lsa, bu to‘g‘ri chiziqlar paralleldir.
3. Agar tekislikda yotmagan to‘g‘ri chiziq tekislikdagi biror to‘g‘ri chiziqqa parallel bo‘lsa, tekislik va to‘g‘ri chiziqlar o‘zaro paralleldir.
4. Agar tekislikka tushirilgan og‘ma tekislikda yotuvchi to‘g‘ri chiziqqa perpendikulyar bo‘lsa, uning proeksiyasi ham og‘ma to‘g‘ri chiziqqa perpendikulyar bo‘ladi.
5. Agar tekislikdagi to‘g‘ri chiziq tekislikka tushirilgan og‘maga perpendikulyar bo‘lsa, bu to‘g‘ri chiziq og‘maning proyeksiyasiga ham perpendikulyar bo‘ladi.
6. Agar ikki to‘g‘ri chiziq uchinchi to‘g‘ri chiziqqa parallel bo‘lsa, ular o‘zaro paralleldir.
7. Agar fazoda ikki to‘g‘ri chiziq uchinchi to‘g‘ri chiziqqa parallel bo‘lsa, ular o‘zaro paralleldir .
8. Agar bir tekislikda yotgan ikki to‘g‘ri chiziq ikkinchi tekislikda yotgan ikki to‘g‘ri chiziqqa mos ravishda parallel bo‘lsa, bu tekisliklar paralleldir.
9. Tekislikda yotuvchi kesishuvchi ikki to‘g‘ri chiziqqa perpendikulyar to‘g‘ri chiziq tekislikka ham perpendikulyardir.

10. Tekislikda yotgan to‘g‘ri chiziq og‘maning proeksiyasiga perpendikulyar bo‘lsa, og‘maning o‘ziga ham perpendikulyardir.
11. Tekislikda og‘maning asosidan uning proeksiyasiga perpendikulyar qilib o‘tkazilgan to‘g‘ri chiziq og‘maning o‘ziga ham perpendikulyardir.
12. Tekislikdagi kesishuvchi ikki to‘g‘ri chiziqqa perpendikulyar bo‘lgan to‘g‘ri chiziq tekislikka ham perpendikulyar.
13. Ikkita parallel tekislikni uchinchi tekislik bilan kesganda hosil bo‘lgan to‘g‘ri chiziqlar o‘zaro paralleldir.
14. Agar tekislik parallel tekisliklardan biriga perpendikulyar bo‘lsa bu tekislik ikkinchi tekislikka ham perpendikulyardir.
15. Fazodagi ikki to‘g‘ri chiziq uchinchi to‘g‘ri perpendikulyar bo‘lsa ular o‘zaro paralleldir.
16. Og‘ma va uning tekislikdagi proeksiyasi orasidagi burchaklaridan eng kichigiga og‘ma va tekislik orasidagi burchak deyiladi.

UCHBURCHAKKA OID BA’ZI MULOHAZALAR

1. Agar ikkita teng yonli uchburchakning asoslari va asoslaridagi burchaklari teng bo‘lsa, bu uchburchaklar tengdir.
2. Agar ikkita teng tomonli uchburchaklarning balandliklari teng bo‘lsa, uchburchaklar tengdir.
3. Agar bir uchburchakning uchta tomoni ikkinchi uchburchakning uchta tomoniga mos ravishda teng bo‘lsa, bu uchburchaklar tengdir.
4. Ikkita to‘g‘ri burchakli uchvurchakning gippotenuzalari va bittadan o‘tkir burchaklari bir-biriga teng bo‘lsa bu uchburchaklar tengdir.
5. Asosi va uchidagi burchagi o‘zaro teng bo‘lgan teng yonli uchvurchaklar o‘zaro tengdir.
6. Teng tomonli uchburchakning balandliklari (medianalari, bissektrisalari) kesishish nuqtasida 2:1 nisbatda bo‘linadi.
7. Muntazam uchburchak medianasining uchdan bir qismi unga ichki chizilgan aylana radiusi bo‘ladi.
8. Muntazam oltiburchakning katta diagonali unga tashqi chizilgan aylananing diametriga teng.

GEOMETRIK MASALALARGA YORDAM

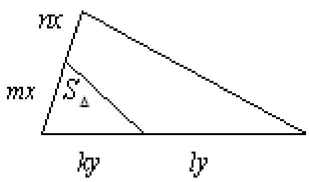
a) Planimetriya

1) hech qaysi uchtasi bir to’g’ri chiziqda yotmaydigan n ta nuqtadan o’tkazish mumkin bo’lgan to’g’ri chiziqlar (kesmalar) soni:

$$\frac{n(n-1)}{2}$$

2) uchburchakning ikki tomoni uning uchidan boshlab hisoblaganda $m:n$ va $k:l$ nisbatda bo’lganda,

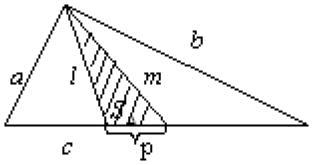
S_Δ -umumiy katta uchburchakning yuzi



$$\frac{S_\Delta}{S}=\frac{mk}{(m+n)(k+l)}$$

$$S_\Delta=\frac{mk}{(m+n)(k+l)}S$$

3) Tomonlari a, b va c bo'lgan uchburchakning c tomoniga tushirilgan mediana va bissektisasi orasidagi yuzani hamda c tomondan ajralgan kesma uzunligini hisoblang.



$$ax+bx=c;\ x=\frac{c}{a+b}.\ p=\frac{|a-b|x}{2}=\frac{|a-b|\cdot c}{2(a+b)};\ S_{\Delta}=\frac{|a-b|}{2(a+b)}\cdot S$$

b) Stereometriyaga doir

o'lchamlari $a\times b\times c$ bo'lgan to'g'ri burchakli parallelepiped ichiga tomoni d bo'lgan kublardan eng ko'pi bilan nechta joylashtirish mumkin.

$$\left[\frac{a}{d}\right]\cdot\left[\frac{b}{d}\right]\cdot\left[\frac{c}{d}\right].$$

BA'ZI TURDAGI TENGLAMA VA TENGSIZLIKLARNI YECHISH SXEMASI

Bir tomoni modulli bo'lgan tenglama va tengsizliklar

$\lvert f\rvert<\varphi$	$\lvert f\rvert\le\varphi$	$\lvert f\rvert=\varphi$	$\lvert f\rvert\ge\varphi$	$\lvert f\rvert>\varphi$
$-\varphi<f<\varphi$ yoki $\begin{cases}-\varphi<f\\ f<\varphi\end{cases}$	$-\varphi\le f\le\varphi$ yoki $\begin{cases}-\varphi\le f\\ f\le\varphi\end{cases}$	$\begin{cases}\varphi\ge 0\\ f=-\varphi,\ f=\varphi\end{cases}$ yoki $\begin{cases}f<0\\ f=-\varphi\end{cases}\cup\begin{cases}f\ge 0\\ f=\varphi\end{cases}$	$f\le-\varphi,\ f\ge\varphi$	$f<-\varphi,\ f>\varphi$

Ikkala tomoni ham modulli bo'lgan tenglama va tengsizliklar

$\lvert f\rvert<\lvert\varphi\rvert$	$\lvert f\rvert\le\lvert\varphi\rvert$	$\lvert f\rvert=\lvert\varphi\rvert$	$\lvert f\rvert\ge\lvert\varphi\rvert$	$\lvert f\rvert>\lvert\varphi\rvert$
$f^2<\varphi^2$ yoki $(f-\varphi)(f+\varphi)<0$	$f^2\le\varphi^2$ yoki $(f-\varphi)(f+\varphi)\le 0$	$f=-\varphi$ va $f=\varphi$	$f^2\ge\varphi^2$ yoki $(f-\varphi)(f+\varphi)\ge 0$	$f^2>\varphi^2$ yoki $(f-\varphi)(f+\varphi)>0$

Bir tomoni kvadrat ildizli bo'lgan tenglama va tengsizliklar

$\sqrt{f}<\varphi$	$\sqrt{f}\le\varphi$	$\sqrt{f}=\varphi$	$\sqrt{f}\ge\varphi$	$\sqrt{f}>\varphi$
$\begin{cases}\varphi>0\\ f\ge 0\\ f<\varphi^2\end{cases}$	$\begin{cases}\varphi\ge 0\\ f\ge 0\\ f\le\varphi^2\end{cases}$	$\begin{cases}\varphi\ge 0\\ f=\varphi^2\end{cases}$	$\begin{cases}\varphi<0\\ f\ge 0\end{cases}\cup\begin{cases}\varphi\ge 0\\ f\ge\varphi^2\end{cases}$	$\begin{cases}\varphi<0\\ f\ge 0\end{cases}\cup\begin{cases}\varphi\ge 0\\ f>\varphi^2\end{cases}$

Ikkala tomoni ham kvadrat ildizda bo'lgan tenglama va tengsizliklar

$\sqrt{f}<\sqrt{\varphi}$	$\lvert f\rvert\le\sqrt{\varphi}$	$\lvert f\rvert=\sqrt{\varphi}$	$\sqrt{f}\ge\sqrt{\varphi}$	$\sqrt{f}>\sqrt{\varphi}$
$0\le f<\varphi$ yoki $\begin{cases}f\ge 0\\ f<\varphi\end{cases}$	$0\le f\le\varphi$ yoki $\begin{cases}f\ge 0\\ f\le\varphi\end{cases}$	$\begin{cases}f=\varphi\\ f\ge 0\end{cases}$ yoki $\begin{cases}f=\varphi\\ \varphi\ge 0\end{cases}$	$f\le\varphi\ge 0$ yoki $\begin{cases}f\ge\varphi\\ \varphi\ge 0\end{cases}$	$f>\varphi\ge 0$ yoki $\begin{cases}f>\varphi\\ \varphi\ge 0\end{cases}$

Logarfmik tenglama va tengsizliklarni yechish sxemasi.

$\log_{\varphi} f<0$	$\log_{\varphi} f\le 0$	$\log_{\varphi} f=0$	$\log_{\varphi} f\ge 0$	$\log_{\varphi} f>0$

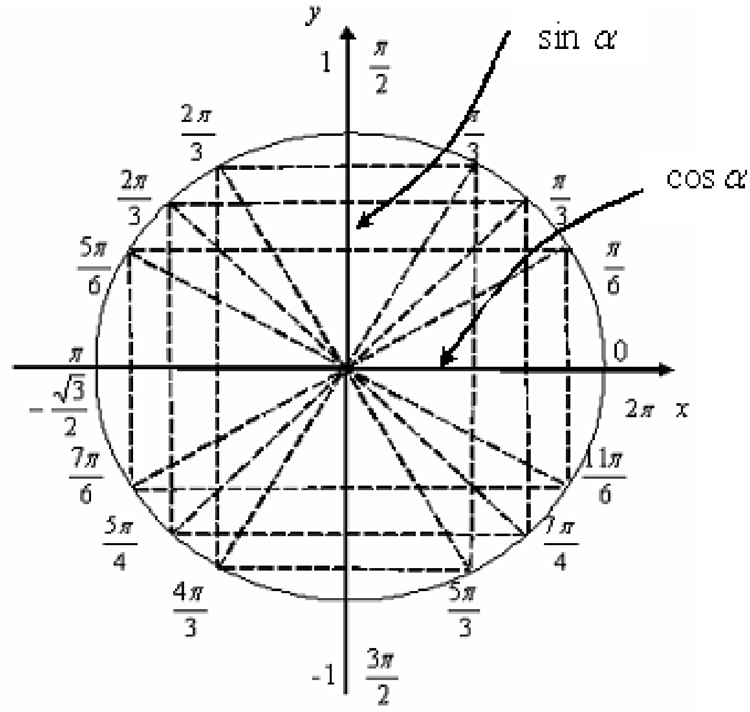
$\begin{cases}(f-1)(\varphi-1)<0\\ f>0\\ \varphi>0,\\ \varphi\neq 1\end{cases}$	$\begin{cases}(f-1)(\varphi-1)\leq 0\\ f>0\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}f=1\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}(f-1)(\varphi-1)\geq 0\\ f>0\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}(f-1)(\varphi-1)>0\\ f>0\\ \varphi>0\\ \varphi\neq 1\end{cases}$

$\log_\varphi f_1<\log_\varphi f_2$	$\log_\varphi f_1\leq\log_\varphi f_2$	$\log_\varphi f_1=\log_\varphi f_2$	$\log_\varphi f_1\geq\log_\varphi f_2$	$\log_\varphi f_1>\log_\varphi f_2$
$\begin{cases}(f_1-f_2)(\varphi-1)<0\\ f_1>0\\ f_2>0\\ \varphi>0\\ \varphi\neq 1\end{cases}$	$\begin{cases}(f_1-f_2)(\varphi-1)\leq 0\\ f_1>0\\ f_2>0\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}f_1=f_2\\ f_1>0\\ \varphi>0,\varphi\neq 1\end{cases}$ yoki $\begin{cases}f_1=f_2\\ f_2>0\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}(f_1-f_2)(\varphi-1)\geq 0\\ f_1>0\\ f_2>0\\ \varphi>0,\varphi\neq 1\end{cases}$	$\begin{cases}(f_1-f_2)(\varphi-1)>0\\ f_1>0\\ f_2>0\\ \varphi>0\\ \varphi\neq 1\end{cases}$

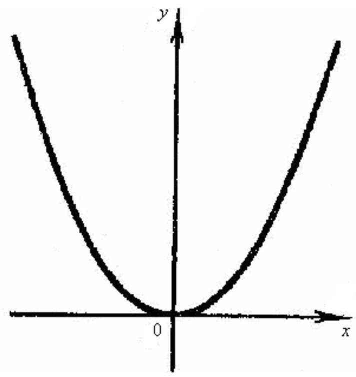
Qo'shimcha

$[f(x)]^{g(x)}=[f(x)]^{\varphi(x)}$	$[f(x)]^{g(x)}\geq[f(x)]^{\varphi(x)}$	$[f(x)]^{g(x)}\leq[f(x)]^{\varphi(x)}$
1) $\begin{cases}g(x)=\varphi(x)\\ f(x)\neq 0\end{cases}$ 2) $\begin{cases}f(x)=0\\ g(x)>0\\ \varphi(x)>0\end{cases}$ 3) $\lvert f(x)\rvert=1$	$\begin{cases}0<f(x)<1\\ g(x)\leq\varphi(x)\end{cases}\cup\begin{cases}f(x)>1\\ g(x)\geq\varphi(x)\end{cases}$	$\begin{cases}0<f(x)<1\\ g(x)\geq\varphi(x)\end{cases}\cup\begin{cases}f(x)>1\\ g(x)\geq\varphi(x)\end{cases}$

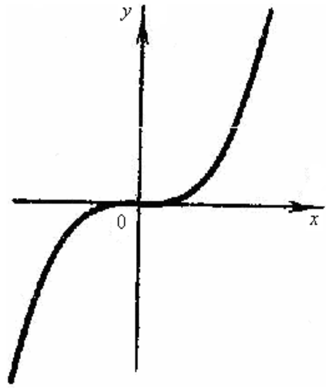
$\sin\alpha$ va $\cos\alpha$ ning ba'zi burchaklardagi qiymatlarini birlik aylanada aniqlanishi



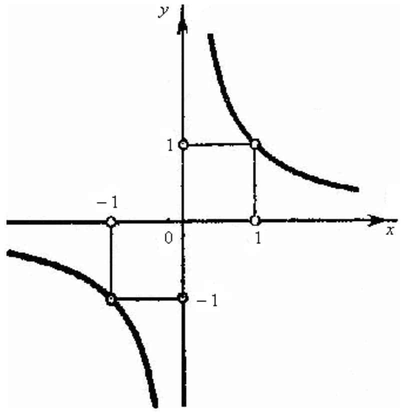
Ba'zi funksiyalar grafiklari



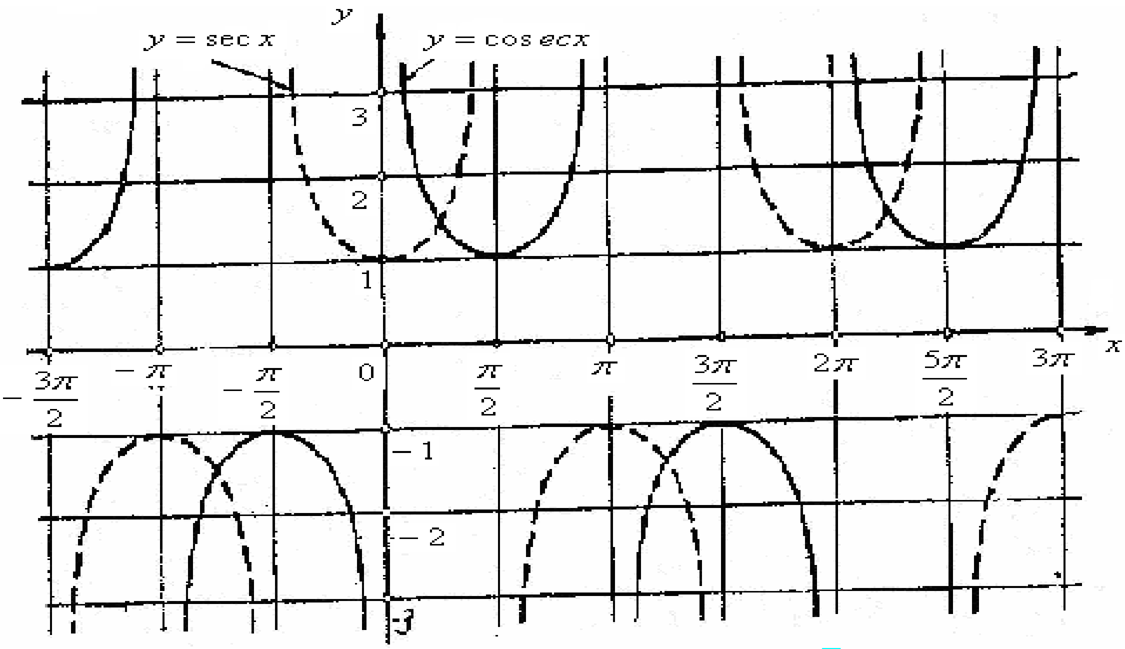
$y = x^2$ parabola.



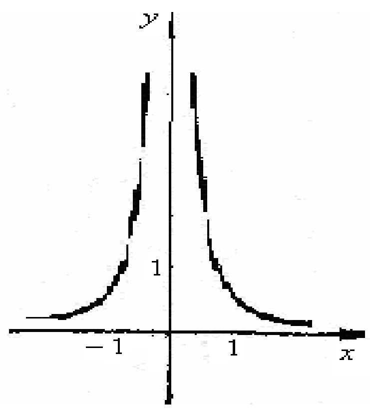
$y = x^3$ kubik parabola.



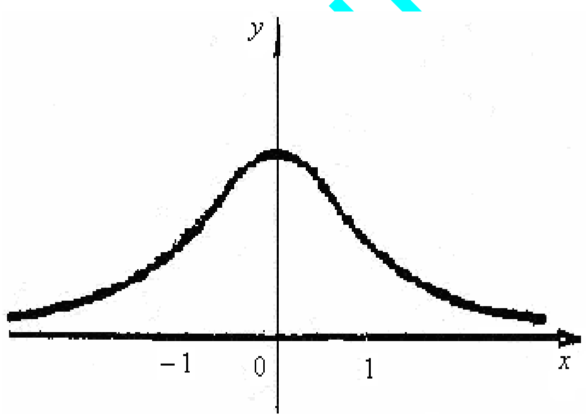
$y = \frac{1}{x}$ giperbola.



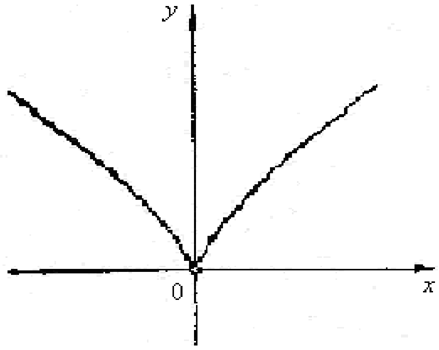
$y = \sec x$ va $y = \operatorname{cos}ec x$ funksiya grafiklari

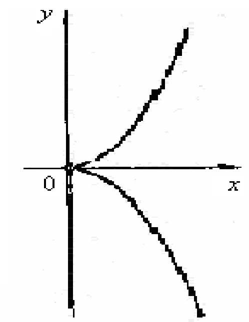


$y = \frac{1}{x^2}$ kasr funksiya grafigi

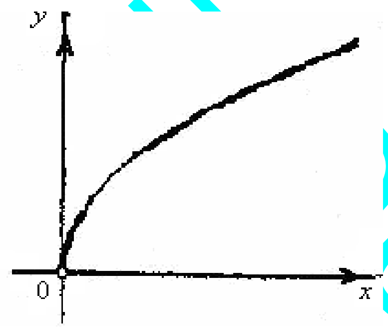


$y = \frac{1}{1+x^2}$ Anyezi zulfi

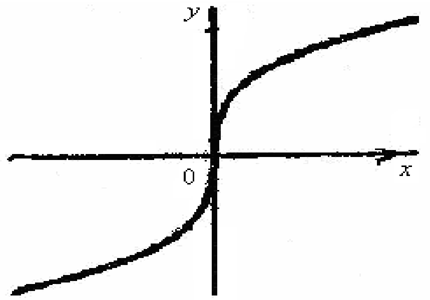




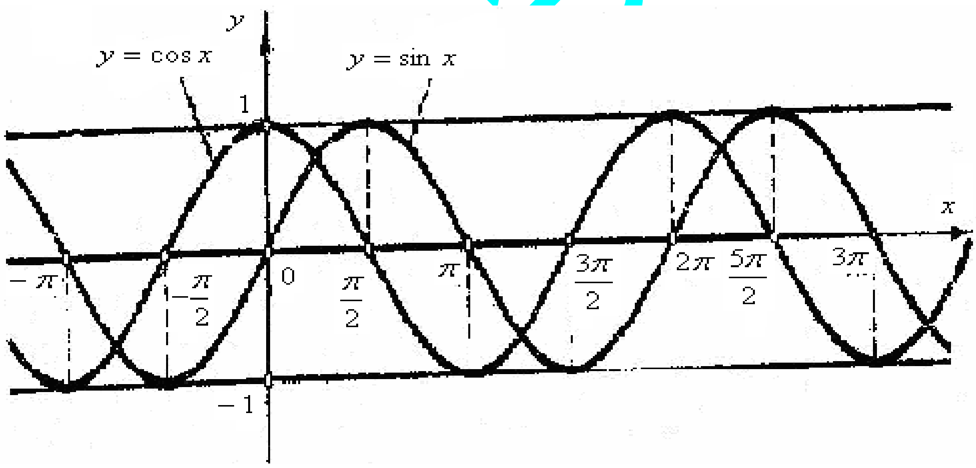
$y = x^{\frac{2}{3}}$ bunda $\begin{cases} x = t^2 \\ y = t^2 \end{cases}$ Nelya parabolasi. $y^2 = x^3$ bunda $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$ yarim kubik parabola



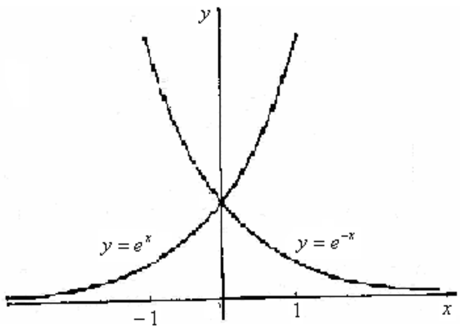
$y = \sqrt{x}$ parabola (yuqori bo'lagi)

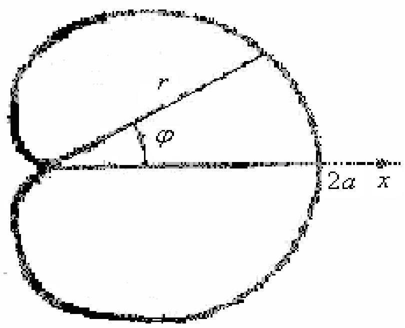


$y = \sqrt[3]{x}$ kubik parabola

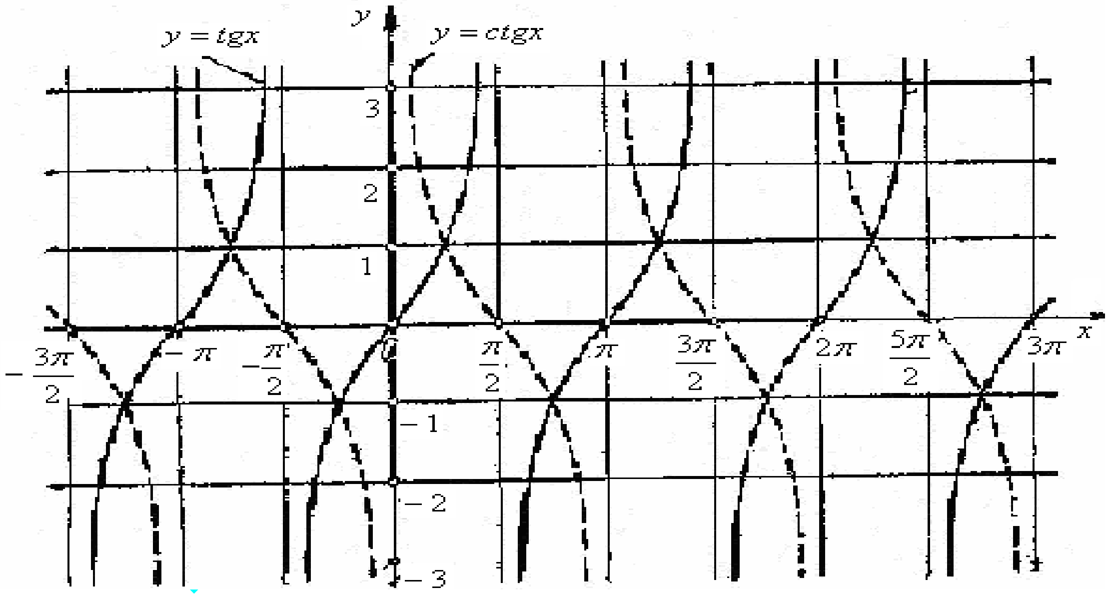


$y = \cos x$ va $y = \sin x$ kosinusoida va sinusoida

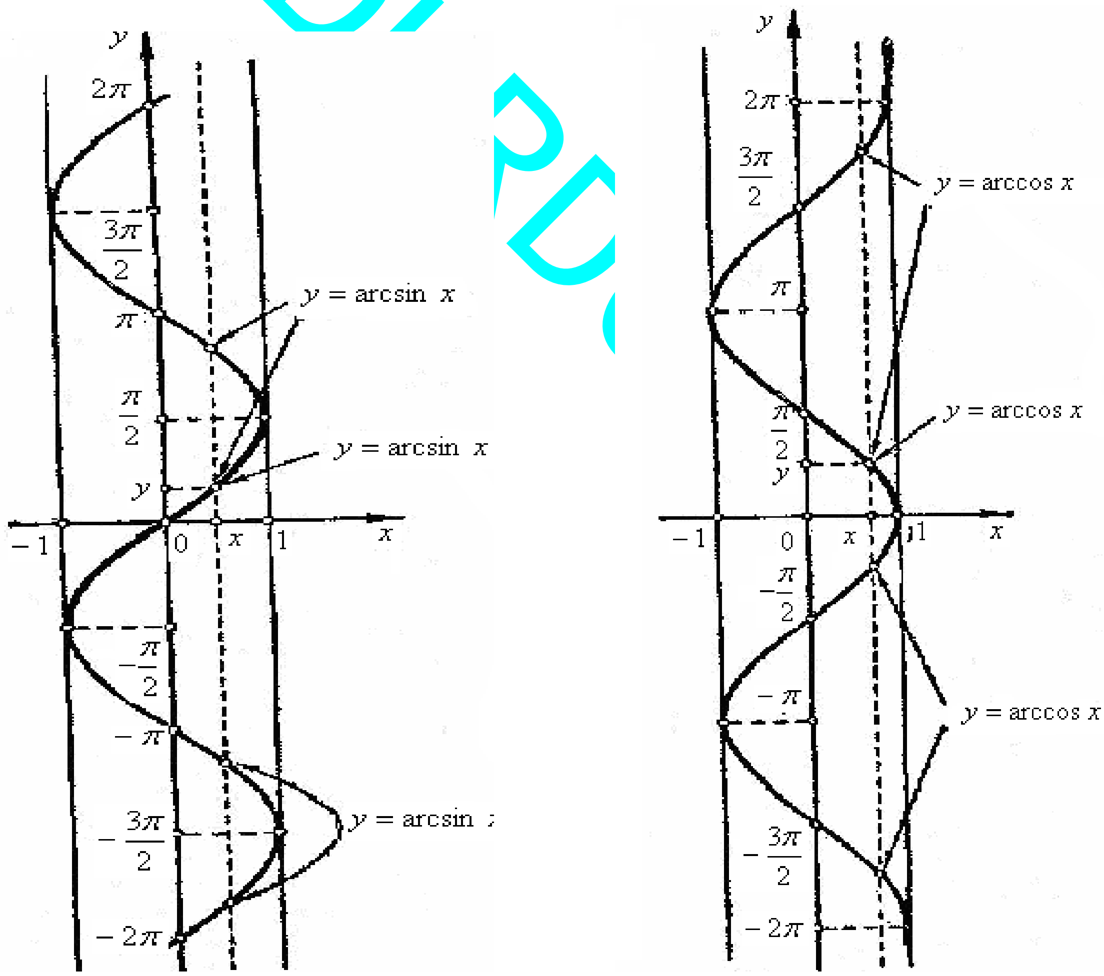




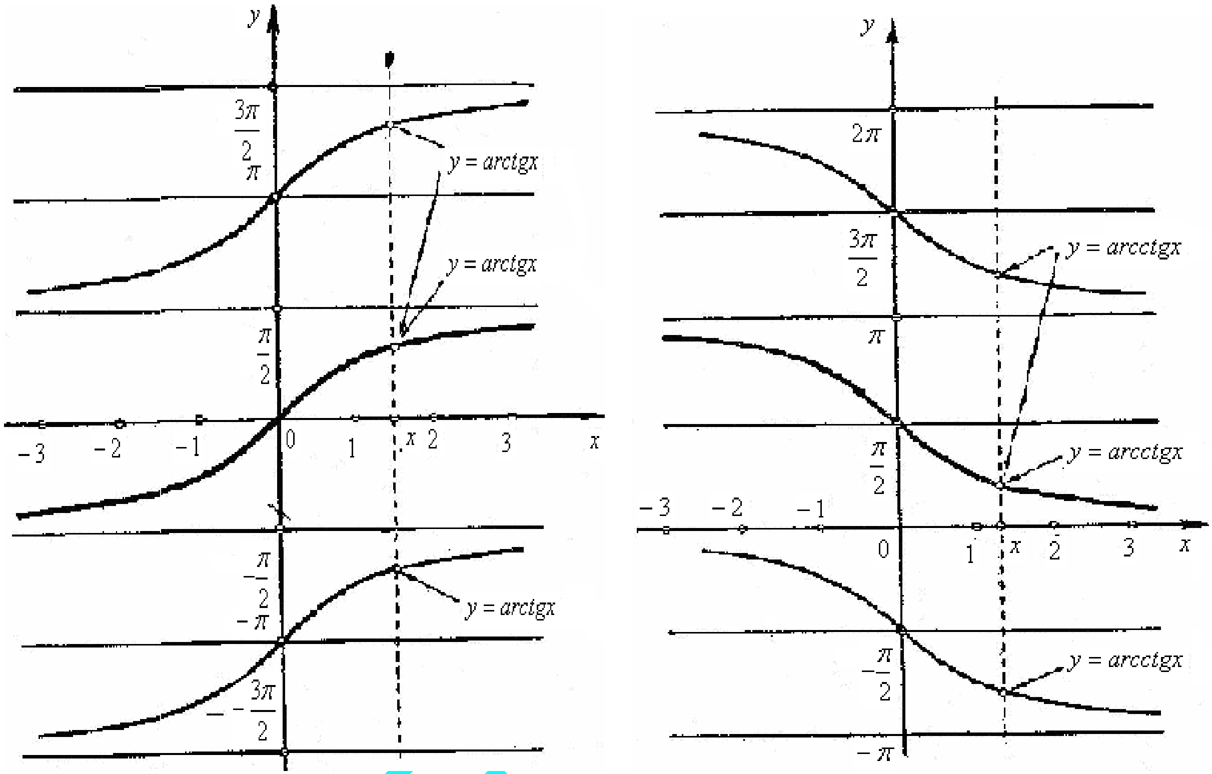
$y = e^x$ va $y = e^{-x}$ ko'rsatkichli funksiyalar grafigi $r = a(1 + \cos\varphi)$ kardioida



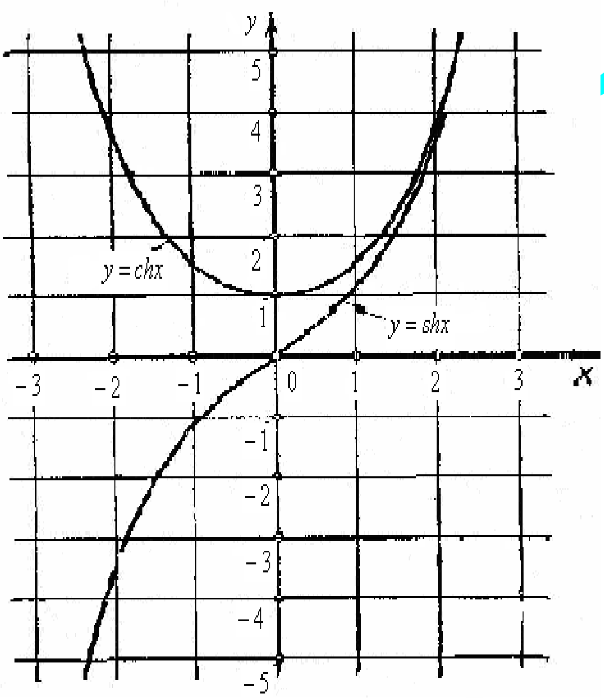
$y = tgx$ va $y = ctgx$ tangesoida va kotangensoida



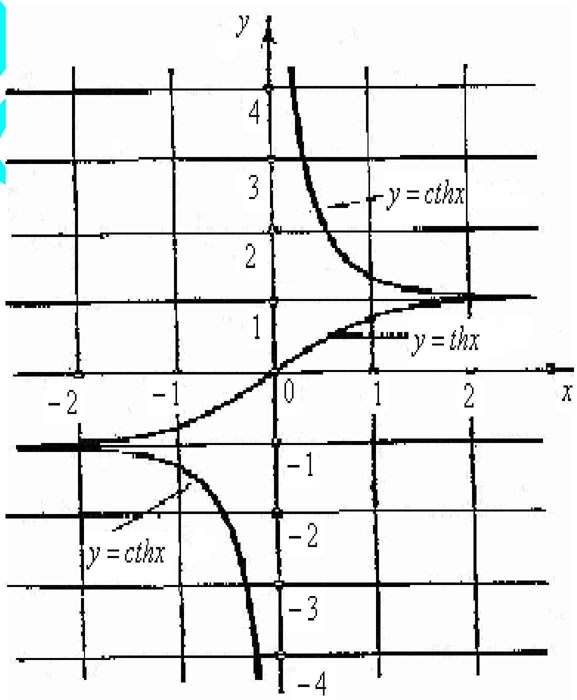
$y = \arcsin x$ va $y = \arccos x$ teskari trigonometrik funksiyalar grafiklari



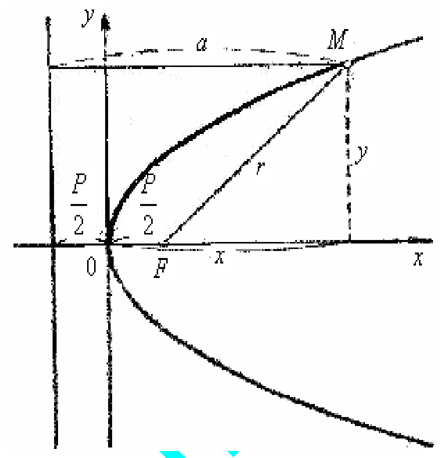
$y = arctgx$ va $y = arcctgx$ teskari trigonometrik funksiyalar grafiklari

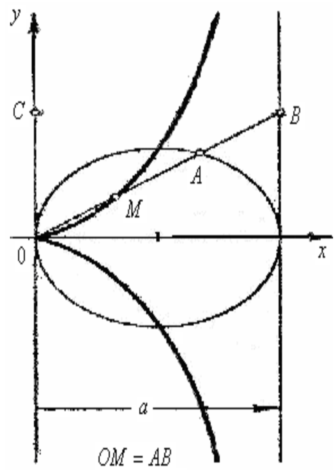


$y = chx$ va $y = shx$ giperbolik funksiyalar
grafiklari



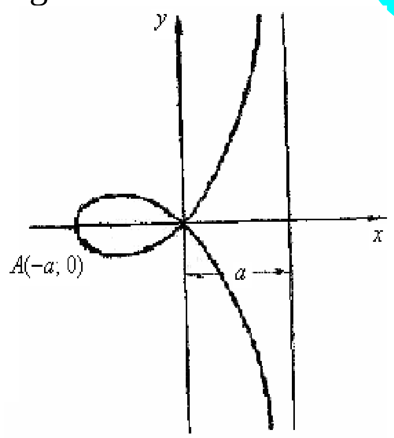
$y = cthx$ va $y = thx$ giperbolik funksiyalar
grafiklari

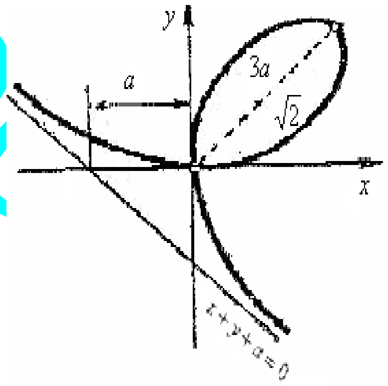




$y^2 = 2px$ parabola

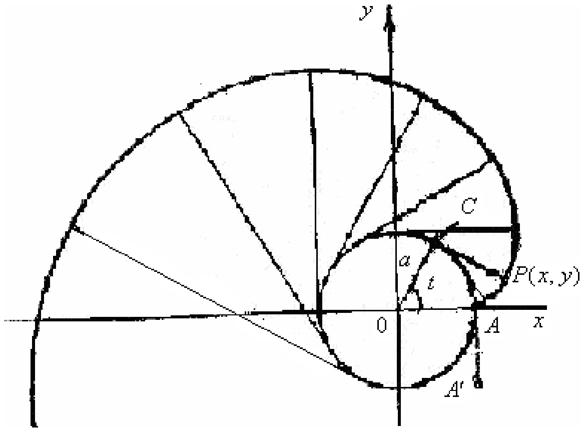
$y^2 = \dfrac{x^3}{a-x}$ bunda $\begin{cases} x = \dfrac{at^2}{1+t^2} \\ y = \dfrac{at^3}{1+t^2} \end{cases}$ Diokles sigsoidasi

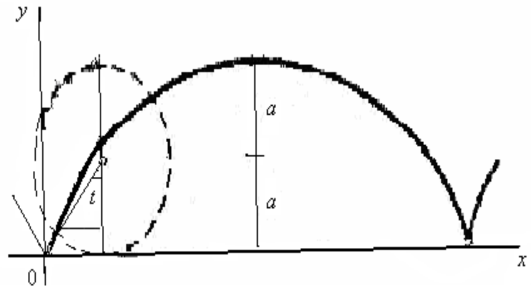




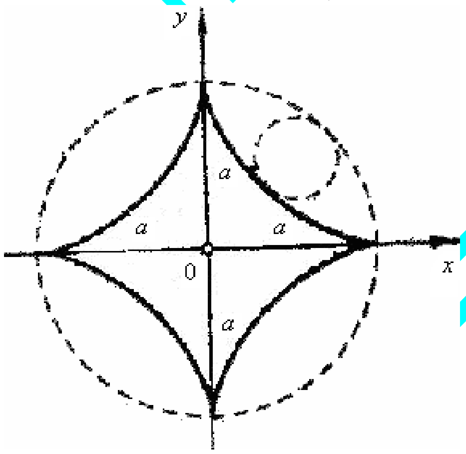
$y^2 = x^2 \dfrac{a+x}{a-x}$ Strofoida

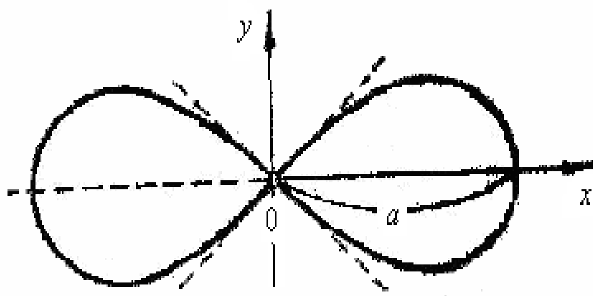
$x^2 + y^2 - 3axy = 0$ bunda $\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$ Dekart yaprog'i





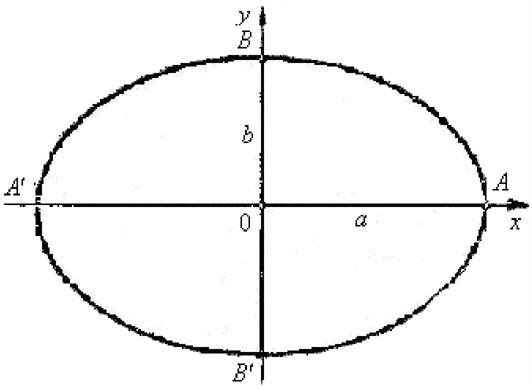
Aylana evolventasi $\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$ $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$ sikloida

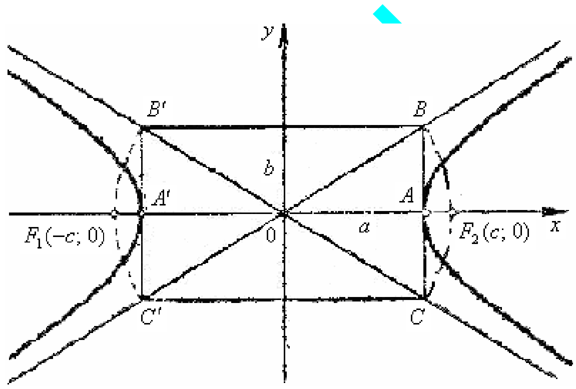




$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$ bunda $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ Giposikloida (astroida)

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$ bunda $r^2 = a^2\cos 2\varphi$ Bernulli Lemniskatasi





$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ bunda $\begin{cases} x = a\cos t, \\ y = b\sin t \end{cases}$ ellips $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ bunda $\begin{cases} x = acht, \\ y = bsht \end{cases}$ giperbola

Ilovalar

GREK ALFAVITI

$\mathrm{A}\alpha$ -alfa
$\mathrm{B}\beta$ -betta
$\Gamma\gamma$ -gamma
$\Delta\delta$ delta
$\mathrm{E}\varepsilon$ -epsilon
$\mathrm{Z}\zeta$ -dzetta
$\mathrm{H}\eta$ -eta
$\Theta\theta$ -teta
$\mathrm{I}\iota$ -iota
$\mathrm{K}\aleph$ -kappa
$\Lambda\lambda$ -lambida
$\mathrm{M}\mu$ -myu
$\mathrm{N}\nu$ -nyu
$\Xi\xi$ -ksi
$\mathrm{O}o$ -omikron
$\Pi\pi$ -pi
$\mathrm{P}\wp$ -po
$\Sigma\sigma$ -sigma
$\mathrm{T}\tau$ -tay
$\Upsilon\upsilon$ -ipsilon
$\Phi\varphi$ -fi
$\mathrm{X}\chi$ -xi
$\Psi\psi$ -psi
$\Omega\omega$ -omega

AYRIM O'ZGARMASLAR

Qiymat	x	$\lg x$	Qiymat	x	$\lg x$
π	3,14159	0,49715	$\frac{1}{e}$	0,36788	1,56571
2π	6,28318	0,79818	e^2	7,38906	0,86859
$\frac{\pi}{2}$	1,57080	0,19612	$\sqrt{e}$	1,64872	0,21715
$\frac{\pi}{4}$	0,78540	1,89509	$\sqrt[3]{e}$	1,39561	0,14176
$\frac{1}{\pi}$	0,31831	1,50285	$M=\lg e$	0,43429	1,63778
π^2	9,86960	0,99430	$\frac{1}{M}=\ln 10$	2,30258	0,36222
$\sqrt{\pi}$	1,77245	0,24857	1 radian	57°17`45``	
$\sqrt[3]{\pi}$	1,46459	0,16572	$arc1°$	0,01745	2,24188
e	2,71828	0,43429	g	9,81	0,99167

TESKARI MIQDORLAR, DARAJA, ILDIZ LOGARIFM

x	$\frac{1}{x}$	x^2	x^3	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	$\lg x$	$\ln x$
1,0	1,000	1,000	1,000	1,000	3,162	1,000	2,154	4,642	0000	0,0000
1,1	0,909	1,210	1,331	1,049	3,317	1,032	2,224	4,791	0414	0,0953
1,2	0,833	1,440	1,728	1,095	3,464	1,063	2,289	4,932	0792	0,1823
1,3	0,769	1,690	2,197	1,140	3,606	1,091	2,351	5,066	1139	0,2624
1,4	0,714	1,960	2,744	1,183	3,742	1,119	2,410	5,192	1461	0,3365
1,5	0,667	2,250	3,375	1,225	3,873	1,145	2,466	5,313	1761	0,4055
1,6	0,625	2,560	4,096	1,265	4,000	1,170	2,520	5,429	2041	0,4700
1,7	0,588	2,890	4,913	1,304	4,123	1,193	2,571	5,540	2304	0,5306
1,8	0,556	3,240	5,832	1,342	4,243	1,216	2,621	5,646	2553	0,5878
1,9	0,526	3,610	6,859	1,378	4,359	1,239	2,668	5,749	2788	0,6419
2,0	0,500	4,000	8,000	1,414	4,472	1,260	2,714	5,848	3010	0,6931
2,1	0,476	4,410	9,261	1,449	4,583	1,281	2,759	5,944	3222	0,7419
2,2	0,454	4,840	10,65	1,483	4,690	1,301	2,802	6,037	3424	0,7885
2,3	0,435	5,290	12,17	1,517	4,796	1,320	2,844	6,127	3617	0,8329
2,4	0,417	5,760	13,82	1,549	4,899	1,339	2,884	6,214	3802	0,8755
2,5	0,400	6,250	15,62	1,581	5,000	1,357	2,924	6,300	3979	0,9163
2,6	0,385	6,760	17,58	1,612	5,099	1,375	2,962	6,383	4150	0,9555
2,7	0,370	7,290	19,68	1,643	5,196	1,392	3,000	6,463	4314	0,9933
2,8	0,357	7,840	21,95	1,673	5,292	1,409	3,037	6,542	4472	1,0296
2,9	0,345	8,410	24,39	1,703	5,385	1,426	3,072	6,619	4624	1,0647
3,0	0,333	9,000	27,00	1,732	5,477	1,442	3,107	6,694	4771	1,0986
3,1	0,323	9,610	29,79	1,761	5,568	1,458	3,141	6,768	4914	1,1314
3,2	0,312	10,24	32,77	1,789	5,657	1,474	3,175	6,840	5051	1,1632
3,3	0,303	10,89	35,94	1,817	5,745	1,489	3,208	6,910	5185	1,1939
3,4	0,294	11,56	39,30	1,844	5,831	1,504	3,240	6,980	5315	1,2238
3,5	0,286	12,25	42,88	1,871	5,916	1,518	3,271	7,047	5441	1,2528
3,6	0,278	12,96	46,66	1,897	6,000	1,533	3,302	7,114	5563	1,2809
3,7	0,270	13,69	50,65	1,924	6,083	1,547	3,332	7,179	5682	1,3083
3,8	00,263	14,41	54,87	1,949	6,164	1,560	3,362	7,243	5798	1,3350
3,9	0,256	15,21	59,32	1,975	6,245	1,574	3,391	7,306	5911	1,3610
4,0	0,250	16,00	64,00	2,000	6,325	1,587	3,420	7,368	6021	1,3863
4,1	0,244	16,81	68,92	2,025	6,403	1,601	3,448	7,429	6128	1,4110
4,2	0,238	17,64	74,09	2,049	6,481	1,613	3,476	7,489	6232	1,4351
4,3	0,233	18,49	79,51	2,074	6,557	1,626	3,503	7,548	6335	1,4586
4,4	0,227	19,36	85,18	2,098	6,633	1,639	3,530	7,606	6435	1,4816
4,5	0,222	20,25	91,12	2,121	6,708	1,651	3,557	7,663	6532	1,5041
4,6	0,217	21,16	97,34	2,145	6,782	1,663	3,583	7,719	6628	1,5261
4,7	0,213	22,09	103,8	2,168	6,856	1,675	3,609	7,775	6721	1,5476
4,8	0,208	23,04	110,6	2,191	6,928	1,687	3,634	7,830	6812	1,5686
4,9	0,204	24,01	117,6	2,214	7,000	1,698	3,659	7,884	6902	1,5892
5,0	0,200	25,00	125,0	2,236	7,071	1,710	3,684	7,937	6990	1,6094
5,1	0,196	26,01	132,7	2,258	7,141	1,721	3,708	7,990	7076	1,6202
5,2	0,192	27,04	140,6	2,280	7,211	1,732	3,733	8,041	7160	1,6487
5,3	0,189	28,09	148,9	2,302	7,280	1,744	3,756	8,093	7243	1,6677
5,4	0,185	29,16	157,5	2,324	7,348	1,754	3,780	8,143	7324	1,6864

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